Contexts as relativized definitions: a formalization via fixed points

Gianni Amati* Fiora Pirri†

October 11, 1997

Abstract

We present a novel account of contexts, formalized as fixed point equations in the modal logic *QKD4Z*. This offers the ability to represent consistency and provability at the object level, with which one can then represent various relationships that must hold between different contexts, such as inheritance, disjointness, compatibility, etc. The logic also offers the ability to name contexts and to obtain explicit sentential representations for these by solving suitable fixed point equations. We illustrate our approach by examples concerning default inheritance of contexts, contradictory contexts, integration of multiple contexts.

1 Introduction

A context is the snapshot of a current state of affairs, to which an intelligent agent relativizes his reasoning. Agent reasoning is local and indexical, in the sense that truth depends on the actual embedded situation in which the agent is or on his specific state of mind. A typical example is when the same term denotes different objects, depending on the situation in which it is uttered. Classical instances of indexicals are words like "I", "here", "this". Generalizing the notion of indexicals, McCarthy argues that contexts give different interpretations to the same term [20] or sentence. For instance, "Spock is the lieutenant of the Enterprise", in the context of Star Treck stories, while it is the name of a dog in the context of the children's book "Look for Spock".

Since McCarthy's Turing Award speech, in 1971, in which contexts have been introduced to AI, some work has been done both to understand all the possible ways contexts are involved in commonsense reasoning (see e.g.[19], Guha's thesis [15], Shoham's discussion [25] and McCarthy's notes [20]) and to reify such a notion into a formal system (see e.g. [17, 6, 13, 12, 3, 5, 7, 10]). In particular, both in [3] and in [10] the $\mathbf{ist}\langle *, * \rangle$ predicate is interpreted as a modal operator and contexts are interpreted in [3] through a multimodal setting and in [10] via fibring. Both the approaches are quite sophisticated: they provide a sound and complete calculus and lift the formalization to first order logic. Applications of contexts have been addressed in [16, 15, 11].

One aspect that, so far, has not been taken into account is how self reference and definitions are relevant in the formalization of contexts. That is, a context, as the snapshot of a current state of affairs, provides a relative, or local, meaning to a set of sentences. In this sense we talk about "contextual sentences" instead of theories denoted by contexts. The difference is subtle. When contexts are seen as given theories, with their own signature (as in Lifschitz [17]), we still have all the limitations of first order logic, we cannot say that a sentence is provable in a context and its

*Fondazione Ugo Bordoni, via Baldassarre Castiglione, 59, 00142 Roma, Italia. e-mail: `gba@fub.it`

†Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza", via Salaria, 113, 00198 Roma, Italia e-mail: `pirri@assi.dis.uniroma1.it`

negation is provable in another context and yet relate the two contexts, without avoiding trivial results. If sentences are contextualized, instead, this is possible. Therefore a contextual sentence is one that refers to it self under certain circumstances. An important reason for which contexts are useful in AI, as noted by Shoham [25], is "economy of representation" and "economy of reasoning". In fact, human reasoning is contextual since shifting from a context to another allows one to forget about redundant or meaningless – for the current context – information. A context is the snapshot of the current situation. Now, this sort of economy of reasoning and economy of information or representation, should be recovered in the formalization of contexts, somehow. Our intuition is that viewing contexts as definitions, or analogously as self-referential statements, is an interesting way to pursue all the advantages that should arise from their formalization.

In other words, consider the following example. Suppose an agent F has the following contextual KB where 'at-home','heat-on', etc. are contexts and *Pred* and *Consist* are suitable operators or meta-predicates talking about provability and consistency of contexts:

$$\begin{array}{l} Pred(\text{`}\textit{at-home}' \rightarrow (\textit{feel-cold} \rightarrow \textit{switch-on-the-heating})) \\ \quad \wedge Consist(\text{`}\textit{at-home}') \\ Pred(\text{`}\textit{heat-on}' \rightarrow (\textit{feel-cold} \rightarrow \textit{increase-the-heat})) \\ \quad \wedge Consist(\text{`}\textit{heat-on}') \\ Pred(\text{`}\textit{in-the-water}' \rightarrow (\textit{feel-cold} \\ \quad \rightarrow \textit{exit-from-the-water})) \quad \wedge Consist(\text{`}\textit{in-the-water}') \\ \neg Consist(\text{`}\textit{at-home}' \wedge \text{`}\textit{in-the-water}') \end{array} \tag{1}$$

Suppose, now, that the heat is on, but F feels cold. Then we add to (1) the following:

$$Pred(\text{`}\textit{heat-on}' \rightarrow \textit{feel-cold}) \wedge Consist(\text{`}\textit{heat-on}') \tag{2}$$

One way to compute *increase-the-heat*, is to provide a sound and complete calculus for contexts. Now, if each denotation, of a context name, is a theory (see e.g. [17]) then a deductive mechanism has to be activated within the theory, named by the specific context. However, if we think thoroughly, we do not really want to know what the theory is about, we want just to know whether, e.g., *increase-the-heat* is the right thing to do in the current context, forgetting about everything else. Agent F needs the minimal information necessary to do the right thing. Therefore, under the economy view, we need to determine which are the contexts *necessary* to get *increase-the-heat* and verify if they are, indeed, *sufficient*. In other words we want to obtain a self-referential statement as follows: "*increase-the-heat* is the solution to the following equation":

$$(2) \wedge (1) \equiv Pred(\textit{increase-the-heat}) \tag{3}$$

whenever all the consistent contexts are suitably substituted for $Pred(\textit{increase-the-heat})$.

In other words, the right answer, or consequence, of a contextual form of reasoning is just the result of suitable substitutions for contexts. In the above example, we do not need the context "at-home", since *heat-on* can depend on the fact that she is in the car or somewhere else. We just need the minimal amount of information. Analogously we do not need the context "in the water". The only necessary and sufficient context is "heat-on". This self referential view of contexts is not new, both Smullyan and Perlis in their papers on self reference [26] and [21] refer to contexts.

Now, (3) make sense if their definitional form is, indeed, a fixed point equation, and thus they are logical truths in some logical system. The only way to ensure this is to show that, for each context, there is at least a fixed point equation that satisfies it. That is, (3) is a theorem of a *logic for contexts*. In this paper we extend the results on definability of concepts, described in [1], to the definability of contexts, by showing that a fixed point theorem for one context can be extended to handle multiple contexts, so that we can treat contexts as parameters and nest them, one into the other, as we like. The extension of the results in [1], proposed here, enlightens an interesting point: contextual reasoning is a generalization of nonmonotonic reasoning.

The formalization we propose, in these notes, uses modal logic. The modal embedding is a necessary condition to handle correctly the fixed point theorem, together with a suitable representation for the predicate that talks about provability and consistency of formulae into each context. However one can freely disregard modal logic and apply the procedures to compute contexts, suggested in the following sections, into classical logic, without any problem.

Our perspective is to interpret a set of axioms as a system of equations. Axioms tie down the use or meaning of those fundamental ideas (or primitive terms, or concepts) they define; they determine what can and what cannot be said about these fundamental ideas. Sometimes the axioms are described as '*implicit definitions*' of the concepts they introduce [22]. We treat an axiom as a logical equation defining certain parameters. Then a set of axioms is regarded as a system of logical equations which determines the *unknown* parameters. This system of equations characterizes certain combinations of sentences as admissible, and other as inadmissible, distinguishing them by means of substitutions. We admit only those substitutions that turn an equation into a *theorem* of the logic. By means of these equations a definite class of admissible sentences is defined, namely the class of those which satisfy the system. These sentences explicitly define the contexts.

2 Preliminaries

We shall use first-order logic and first-order modal logic, this last in a very simplified way. We assume the reader familiar with these languages; refer to [14] and [8] for details. We assume a quantified modal language whose formulae φ are built by means of boolean connectives, quantifiers and modal operators $\Box$ and $\Diamond$. The sole modal logic we consider is the quantified modal logic $KD4Z$, called $QKD4Z$, axiomatized by the following modal schemata and inference rules:

$$\begin{array}{ll} K. & \Box\varphi \wedge \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\ D. & \Box\varphi \to \Diamond\varphi \\ 4. & \Box\varphi \to \Box\Box\varphi \\ Z & \Box(\Box\varphi \to \varphi) \to (\Diamond\Box\varphi \to \Box\varphi) \\ Nec & \Box\varphi/\varphi \end{array} \tag{4}$$

The semantics of $KD4Z$ is given in [1]. In particular, it is shown that $KD4Z$ is decidable. For the modal logic $QKD4Z$ $\Box$ is interpreted as "it is provable in a decidable fragment of first order logic", so $\Box p$ means "p is classically provable". This interpretation is better understandable when contextual formulae are concerned. By a contextual sentence is meant a sentence of the form $E(c)$ in which a parameter, usually denoted by c, occurs in the scope of a $\Box$. This requirement is akin to the usual reification of a context [20] and allows for substitutions of sentences for parameters. So, if φ is a sentence, $\Box\Diamond(c \wedge \varphi)$ is a contextual sentence. Contextual sentences can be built by parameters, denoting contexts, boolean connectives and first order formulae. The modal operators are used to state that a formula φ is *derivable* from a given context, in which case we write $\Box(c \to \varphi)$ or that it is *consistent* with the context, in which case we write $\Box\Diamond(c \wedge \varphi)$. Therefore the class of first order sentences is extended to a class of sentences containing parameters. Observe that the representation we give is weaker than the $ist(c, p)$ proposed by McCarthy, where the interpretation of ist is "p is *true* in context c". In our formalism we deal with $\Box(c \to p)$, i.e. "p is provable from context c". However an advantage of our modal approach is that we can deal also with a consistency representation: $\Box(c \wedge p)$, i.e. "p is consistent with context c". If $E(c)$ is a contextual sentence then $\Box E(c)$ is a *modal predicate* in the context c. A context is computed by solving a fixed point equation of the form

$$E(c) \equiv c \tag{5}$$

A solution to the above equation is found by suitably substituting a formula for c. A formula $\mathcal{T}$ is a solution of the above equation, iff

$$\models_{QKD4Z} E(\mathcal{T}) \equiv \mathcal{T} \tag{6}$$

We recall here basic results on fixed points given in [1]. A modal predicate $E(c)$, with parameter c, is called an *implicit consistency predicate* whenever $\models_{QKD4Z} E(c) \rightarrow \Box\Diamond c$. In the following the subscript $QKD4Z$ will be dropped.

Theorem 1 Let $E(c)$ be an implicit consistency modal predicate. Then:

1. $\models E(E(\top)) \equiv E(\top)$;
2. $\models E(\bot) \equiv \bot$;
3. Any fixed point $\mathcal{T}$ of $E(c)$ is of the form $E(\top) \wedge \Box\Diamond\gamma$, with γ any formula.

Theorem 1 states that there exist formulae which, substituted for c turn fixed point equation (5) into a theorem of the logic $QKD4Z$; it also specifies what has to be done in order to get these formulae. E.g let $E(c)$ be $\Box(c \rightarrow \alpha) \wedge \Box\Diamond(c \wedge \beta)$ then $\Box\alpha \wedge \Box\Diamond\beta$ is a *fixed point solution*, that is, $\Box\alpha \wedge \Box\Diamond\beta$ substituted for c satisfies (6). In this sense we can say that the formula, which is a solution to the fixed point equation, *realizes* a context, because it gives an explicit representation of the context, previously designated by a parameter.

When multiple contexts are used, these will be indexed yielding sentences of the form $E(c_1, \ldots, c_n)$. The computation of several contexts is achieved by solving a system of fixed point equations. This is the next topic.

3 Computing multiple contexts

Now, we present a general formalism for handling contexts, extending the results of [1] for single contexts. Multiple contexts are used to represent different states of affairs: to show that different information sources are conflicting or alternative, to state that two situations are contradictory, to complete the information at hand, to specify sub-context inheritance, to indicate that a term (like "now") changes its value according to the context, etc. To express such a variety of uses of contexts, we have extended the results from [1], to manage many parameters, each designating a formula. These are computed by solving a system of equations, instead of a single equation.

The novelty here, having to do with multiple contexts, is that relations among contexts have to be declared. For example "Bill" is the name of the US president, and "bill" is a bird's beak; to maintain the difference we require two contexts, say c_1 and c_2. However we also need to declare what is the relationship between these two contexts: $\Box(c_1 \equiv \neg c_2)$ means that the two contexts can never be chosen together; $\Box\Diamond(c_1 \wedge c_2)$ states that the two contexts are consistent. The language for multiple contexts is a natural extension of that for a single context: a countable number of parameters – denoting contexts – is added and formulae are built by first order formulae, boolean connectives, parameters and modal operators. We show, now, that multiple contexts can be easily computed. A theory with n contexts can be reduced to a system of n logical equations, which yields a set of solutions for those contexts.

Let $E(c_0, \ldots, c_n)$ be a modal predicate. Let $E_i(c_0, \ldots, c_n)$ be of the form $E(c_0, \ldots, c_n) \wedge \Box\Diamond c_i$, $0 \leq i \leq n$. For each i, $0 \leq i \leq n$, $E_i(c_0, \ldots, c_i)$, is an *implicit consistency modal predicate with respect to the context* c_i.

Theorem 2 Let, for $0 \leq i \leq n$, $E_i(c_0, \ldots, c_i)$ be an implicit consistency modal predicate w.r.t. the context c_i. Then there exists a sentence $\mathcal{T}$ in which only those variables of $E(c_0, \ldots, c_n)$ other than $c_0, \ldots, c_n$ occur and such that, if $\mathcal{T}_i = \mathcal{T}$ or $\mathcal{T}_i = \bot$:

$$\{\models E_i(\mathcal{T}_1, \ldots, \mathcal{T}_i, \ldots \mathcal{T}_n) \equiv \mathcal{T}\}_{1 \leq i \leq n}$$

The next characterization of the structure of fixed points gives an easy way to compute solutions of a system of fixed point equations.

Corollary 1 All the fixed point solutions to the equations $E_i(c_0, \ldots, c_n) = c_i$ are of the form $E(\mathcal{T}_1, \ldots, \mathcal{T}_n) \wedge \Box\Diamond\gamma$, for any γ, where $\mathcal{T}_i$ is either $\top$ or $\bot$.

We solve the system of logical equations as follows: solve one equation with respect to the given context, possibly with the solution $\bot$. A sentence not mentioning the context is obtained. Substitute the solution into the remaining equations and proceed iteratively until no equation is left. By Theorem 1 this is equivalent to simultaneously substituting $\bot$ and $\top$ in $\Box E(c_1, \ldots, c_n)$ and obtaining a solution $\mathcal{T} \wedge \Box\Diamond\gamma$ for arbitrary γ.

Example 1 Let $E(c_1, c_2)$ be of the form:

$$(\Box\Diamond(c_1 \wedge A) \rightarrow \Box(c_1 \rightarrow B)) \wedge (\Box\Diamond(c_2 \wedge \neg A) \rightarrow \Box(c_1 \rightarrow \neg B)) \tag{7}$$

Where A and B are formulae. Then $E(c_1, c_2)$ yields the following system of two equations:

$$\left\{ \begin{array}{l} (\Box\Diamond(c_1 \wedge A) \rightarrow \Box(c_1 \rightarrow B)) \wedge (\Box\Diamond(c_2 \wedge \neg A) \rightarrow \\ \qquad \Box(c_1 \rightarrow \neg B)) \wedge \Box\Diamond c_1 \equiv c_i \\ (\Box\Diamond(c_1 \wedge A) \rightarrow \Box(c_1 \rightarrow B)) \wedge (\Box\Diamond(c_2 \wedge \neg A) \rightarrow \\ \qquad \Box(c_1 \rightarrow \neg B)) \wedge \Box\Diamond c_2 \equiv c_i \end{array} \right. \tag{8}$$

The problem, now, is to find a sentence $\mathcal{T}$ such that, for some c_i, $c_i/\mathcal{T}$ satisfies the two equations. Theorem 2 gives us an algorithm to find such a sentence. That is, substituting c_i either with $\top$ or $\bot$. For example, substituting both c_1 and c_2 for $\top$ in $E(c_1, c_2)$, we get:

$$\begin{array}{l} (\Box\Diamond(\top \wedge A) \rightarrow \Box(\top \rightarrow B)) \wedge (\Box\Diamond(\top \wedge \neg A) \rightarrow \\ \qquad \Box(\top \rightarrow \neg B)) \wedge \Box\Diamond\top \end{array}$$

The above formula is simplified as follows:

$$(\Box\Diamond A \rightarrow \Box B) \wedge (\Box\Diamond\neg A \rightarrow \Box\neg B) \tag{9}$$

Let (9) be $\mathcal{T}$, then, by Theorem 1, $\mathcal{T} \wedge \Box\Diamond\gamma$ is a solution to the system (8) for any γ. In particular we have two solutions to (8), one taking $\Box\Diamond A$ for $\Box\Diamond\gamma$ and the other taking $\Box\Diamond\neg A$. Intuitively (7) says that "if A is consistent with c_1 then B is derivable from c_1, and if $\neg A$ is consistent with c_2 then $\neg B$ is derivable from c_1". Therefore choosing $\Box\Diamond A$ we make c_1 active, while choosing $\Box\Diamond\neg A$ we make c_2 active. In particular (8) has 5 solutions: $E(\bot, \bot) = \bot$, $E(\bot, \top) \equiv \top$ $E(\top, \bot) \equiv \Box\neg B$ and, finally, $E(\top, \top)$ has two fixed points, $\Box B$ and $\Box\neg B$.

We have thus proved that either $\Box B$ or $\Box\neg B$ are provable from the contexts. Observe that the trick of substituting the c_i for $\top$ and $\bot$, in view of the above theorems, is the most intuitive way to compute the fixed point, that is, to compute the formula which, substituted for the contexts yields a fixed point solution to the system of equations.
To recover the classical theories from the modal solutions to the contextualized theories, we simply form, for each fixed point solution $\mathcal{T}_i$, the theory $\Delta_i = \{\varphi \mid \mathcal{T}_i \models \Box\varphi\}$.

The upshot of the fixed point approach to contextual reasoning is, thus, the following. To know whether a formula is relatively provable w.r.t. a certain context we have to show that such a formula is a solution to the system of equations in which many contexts are considered. On the other hand, to know what is provable from some contexts, substitute $\bot$ and $\top$ for the context parameters and find the fixed point solutions. Each solution tells us which contextual sentence is relatively provable w.r.t. each context. This way of reasoning is, thus, very simple because showing provability of a sentence from a contextualized theory amounts just to performing substitutions, exploiting the definitional pattern of the fixed point equation. While showing what kind of sentences the context is naming amounts to performing substitutions for $\top$ and $\bot$ thus getting the sentences which are solutions to the system of equations.

4 Contextualization

Our point of view is, therefore, that contexts are *relativistic points of view*, that is, we do not think of a context as a given theory from which a sentences has to be derived or in which a sentence is true. We shall come back on the difference between viewing contexts as 'predefined theories' and our approach, in the last section of these notes. Our relativistic point of view makes contextual reasoning 'essential', that is, the context is used to tell us whether the sentences, that we are going to interpret contextually, are provable or consistent in the contexts, which is verified whenever these sentences are the context themselves, i.e. satisfy the system of equations.

The problem is, now, how do we define a contextualized theory; how many contexts should we use; when should we introduce a context; which are the relations we should establish among contexts, etc.? These problems, of course, concern not only our approach but contextual reasoning in general. So far no one has discussed this issue.

With our approach, as we shall see, the problem of the number of contexts affects the combination of substitutions increasing the number of possible solutions, but since contexts are not predefined theories we can easily deal with a great number of contexts, although finite.

The choice of the contextualization is driven by the representation of the specific state of affairs. There are no general criteria for doing this, just as there are no general criteria to represent a certain state of affairs in first-order or modal logic, etc. Despite this, we give a few criteria. Given a classical representation of a certain state of affairs, contextualization involves the construction of new sentences according to the following requirements:

- Declare the set of contexts to which sentences have to be relativized;
- Declare the dependency of sentences on a context. Either a sentence φ must be a consequence of the context c_i, in which case declare $\Box(c_i \to \varphi)$; or it must be defined as a necessary and sufficient condition for the existence of a context c_i, $\Box(c_i \equiv \varphi)$, or it must be consistent with a context c_i, $\Box\Diamond(c_i \wedge \varphi)$;
- Combine the contexts to declare their mutual dependencies.

Observe that one has not to figure out all the potential alternatives before such an approach works as, in fact, the interesting solutions to the system of equations, that is those $\mathcal{T}$ such that $\mathcal{T} \neq \bot$, are independent of the number of contexts used. One context might give n interesting solutions and another only one $\perp$ solution. Therefore declaring the contexts is just a matter of the way sentences need to be relativized, at the most intuitive level of reasoning.

All the above points might be made clearer by discussing a few examples of contextualization.

1. **Alternative choices.** Suppose the representation regards flipping a coin:

$$\begin{array}{l} flip(coin) \to head(coin) \vee tail(coin) \\ head(coin) \to win \\ tail(coin) \to lose \end{array}$$

Clearly, flipping a coin requires at least two contexts, that is: 1. if flipping yields a head, 2. if flipping yields a tail. The action of flipping can be itself seen as a context, for example the outermost context. Let us introduce, therefore the following definitions:

$$\Box(c_0 \equiv flip(coin)), \Box(c_1 \equiv head(coin)), \Box(c_2 \equiv tail(coin))$$

and the following statement saying that the two contexts are mutually exclusive and that flipping a coin is everywhere consistent:

$$\Box\Diamond c_0 \wedge \neg\Box(c_1 \wedge c_2)$$

We now need to say that, in the context of flipping, if head results then win else lose, which is expressed as:

$$\Box(c_0 \to \Box(c_1 \to win)),\ \Box(c_0 \to \Box(c_2 \to lose))$$

The above contextualization, indeed, has two solutions, one with $\Box win$ and $\Box head(coin)$ and the other with $\Box lose$ and $\Box tail(coin)$. Observe that, with this example, we could have achieved a similar result by suitably defining a default theory [23], i.e. a set of defaults of the form $flip(coin) : head(coin)/win \wedge \neg tail(coin)$. And, in fact, default logic offers a form of contextualization, however there are examples – the multiple source, in the next section is one – in which default logic cannot capture multiple contexts., while it always captures a contextualized theory with only one context (see [1])

2. **Contradictory contexts** Consider two sentences in which the same term, *Spock*, designates in one a man who tells the truth and in the other a liar: telling the truth is the negation of being a liar.

$$TellTruth(Spock), Liar(Spock), Liar(Spock) \equiv \neg TellTruth(Spock) \tag{10}$$

We contextualize (10) as follows.

$$1.\quad \Box(c_0 \equiv Liar(Spock)), \Box(c_1 \equiv TellTruth(Spock)) \quad 2.\quad \Box(c_0 \equiv \neg c_1)$$

where 1. defines the two contexts and 2. declares the contradiction between them.

There are two solutions, one with $\Box Liar(Spock), \Box\neg TellTruth(Spock)$ and the other with $\Box TellTruth(Spock)$, $\Box\neg Liar(Spock)$.

Observe that we have not to figure out what the consistent subset are, to get contextualization. Suppose, in fact that we add $Lieutenant(Spock, Enterprise)$, then we can still maintain the two contexts and still get the two alternatives solutions:

$$\begin{array}{ll} 1. & \Box(c_0 \equiv Liar(Spock) \wedge Lieutenant(Spock, Enterprise)), \\ 2. & \Box(c_1 \equiv TellTruth(Spock) \wedge Lieutenant(Spock, Enterprise)) \end{array}$$

3. **Inheritance.** Consider the following example:

1. *Italians go to church on Sunday, love soccer and eat pasta for dinner;*
2. *Neapolitans eat pizza for dinner;*
3. *No one eat pasta and pizza for dinner;*
4. *Neapolitans are Italians;*

We want the context of Naples to inherit what is true in the context of Italy, except for eating pasta for dinner. Let $E(c, c_I, c_N)$ be the conjunction of the following contextual sentences:

$C1.$ $\Box(c \vee c_I \equiv \forall x\ Italian(x)), \Box(c \vee c_I \equiv \forall x\ goesToChurch(x)), \Box(c \vee c_I \equiv \forall x\ lovesSoccer(x))$
$\Box(c_I \to \forall x\ eatPasta(x))$
$C2.$ $\Box(c_N \to \forall x\ (Neapolitan(x) \wedge eatPizza(x))$
$C3.$ $\neg\Box(c_I \wedge c_N), \Box\Diamond(c_I \vee c)$
$C4.$ $\Box\Diamond c_I \to \Box(c_I \equiv Italy), \Box\Diamond c_N \to \Box(c_N \equiv Naples)$

First observe that we introduce three contexts c, c_N and c_I; c_I is the context for Italy, c_N for Naples and c designates their intersection. Now, the first sentence of (C1) says that *Italians* are both the contexts c and c_I. The other sentences of (C1), are a contextualization of (1); i.e. in the context c_I people are Italians, go to church and love soccer. (C2) says that in the context c_N people are Neapolitans and eat pizza. The first sentence of (C3) says that contexts are alternatives and the second is a consistency condition for inheritance. We can interpret it as "either the context of Italy or what it can be inherited from it must be consistent". There are only three consistent solutions to the set of equations $\{\Box E(c, c_I, c_N) \wedge \Box\Diamond c_i\}$: $\Box E(\top, \top, \bot)$, $\Box E(\top, \bot, \top)$, $\Box E(\bot, \top, \bot)$. The first and the third are equivalent. $\Box E(\top, \top, \bot)$ by demodalization, yields the following:

$$\begin{aligned} Italy \equiv\ & \forall x(Italian(x) \wedge eatPasta(x) \wedge \\ & goesToChurch(x) \wedge lovesSoccer(x)) \end{aligned}$$

$\Box E(\top, \bot, \top)$ yields:

$$\begin{aligned} Naples \equiv\ & \forall x(Neapolitan(x) \wedge Italian(x) \wedge eatPizza(x) \\ & \wedge goesToChurch(x) \wedge lovesSoccher(x)) \end{aligned}$$

The example illustrates how *default* inheritance for contexts is handled in our proposal. By default, *sub-contexts inherit entire theories from their super-contexts.* Neapolitans inherit Italian's entire theory, except for eating pasta. In contrast, conventional nonmonotonic default inheritance mechanisms provide only for the inheritance of *properties*, not theories. This ability of contexts to inherit theories is one of the most useful features of our proposal. Next we give one more example on integrating different sources.

5 Integrating information from multiple sources

Integrating information from multiple sources is a key problem in many AI research areas such as multi-databases [2] and multi-reasoning systems [9]. In this section we propose a way of merging different knowledge bases via contextualization. This merging is achieved by suitably contextualizing the set of sentences in each knowledge base; the choice follows the method given in Section 4. Consider the following query:

$$\exists x(NovChar(x, Holmes) \wedge DetRole(x, Holmes)) \wedge \exists y(NovChar(y, Holmes) \wedge InJap(y, Holmes)) \tag{11}$$

It asks for the title of a novel having Holmes as a character, and in which Holmes plays the role of a detective and the title of a novel, taking place in Japan, whose character is also someone named Holmes. The first KB contains the following information:

$$\begin{array}{ll} 1a. & NovChar(\text{“A study in scarlet”}, Holmes), \\ 2a. & \top : AuthOf(ConanDoyle, x) \wedge NovChar(x, Holmes)/DetRole(x, Holmes), \\ 3a. & NovChar(x, Holmes) \rightarrow InLond(x, Holmes); \end{array} \tag{12}$$

These state that normally Holmes is a character of novels written by Conan Doyle, in which he plays the role of a detective, in London. Observe that (2a) is a default. We have, intuitively, two contexts here, one is the context of English Literature and the other is the context of Holmes' stories. Let us call the context of English literature c_1 and that of Holmes' stories c_2.

A suitable contextualization of the above KB yields:

$$\begin{array}{ll} Ca1. & \Box(c_1 \rightarrow NovChar(\text{“a study in scarlet”}, Holmes)), \\ Ca2. & \Box\Diamond(c_1 \wedge AuthOf(ConanDoyle, x) \wedge NovChar(\text{“a study in scarlet”}, Holmes)) \rightarrow \\ & \quad \Box(c_1 \rightarrow DetRole(x, Holmes), \\ Ca3. & \Box(c_2 \rightarrow (NovChar(x, Holmes) \rightarrow InLond(x, Holmes))); \end{array} \tag{13}$$

Observe that the contextualization (Ca2) of the default (2a) follows the same translation given in [1]. Here, however we can embedd the default into a contextual theory. Let the other KB contains the following information:

$$\begin{array}{ll} 1b. & NovChar(\text{“A life in Japan”}, Holmes), \\ 2b. & NovChar(x, Holmes) \rightarrow InJap(x, Holmes), \\ 3b. & \top : AuthOf(Holmes, x)/TeachRole(x, Holmes); \end{array} \tag{14}$$

These specify that Holmes is the author of novels taking place in Japan in which he himself, normally, plays the role of a teacher. Analogously as before we create two contexts here, one is Japanise Literature and the other is the contexts of the stories written by Holmes. We call these two contexts c_3 ad c_4. Contextualization of (14) yields:

$$\begin{array}{ll} Cb1. & \Box(c_3 \rightarrow NovChar(\text{“a life In Japan”}, Holmes)), \\ Cb2. & \Box(c_3 \rightarrow NovChar(x, Holmes) \rightarrow InJap(x, Holmes)), \\ Cb3. & \Box\Diamond(c_4 \wedge AuthOf(Holmes, x)) \rightarrow \Box(c_4 \rightarrow TeachRole(Holmes); \end{array} \tag{15}$$

We also incorporate into the KBs the following:

$$\begin{array}{l} \neg(TeachRole(x, Holmes) \wedge DetRole(x, Holmes)) \\ \neg(InJap(x, Holmes) \wedge InLondon(x, Holmes)) \end{array} \tag{16}$$

Observe that the knowledge bases (11,12,14) together with (16) can be seen as default theories which, taken all together, have no extension at all. On the other hand if the query were addressed to each single KB it would not get an answer. Now, let $E(c_1, c_2, c_3, c_4)$ be the conjunction of (13,15) together with the following formulae:

$$\begin{array}{l} \Box(\exists x NovChar(x, Holmes) \wedge DetRole(x, Holmes), \\ \exists y NovChar(y, Holmes) \wedge InJap(y, Holmes)), \\ \neg\Box(c_4 \wedge \neg c_1) \wedge \neg\Box(c_2 \wedge \neg c_3) \end{array}$$

Where the last sentence is the contextualization of (16). There exists a solution $\mathcal{T}$ to the following system of equations

$$\{E(c_1, c_2, c_3, c_4) \wedge \Box\Diamond c_i \equiv c_i\}_{1 \leq i \leq 4} \tag{17}$$

such that, if $\Delta = \{\varphi \mid \mathcal{T} \models \Box\varphi\}$ then Δ entails, in first order logic:
$\exists x NovChar(x, Holmes) \wedge DetRole(x, Holmes) \wedge \exists y NovChar(y, Holmes) \wedge InJap(y, Holmes)$

The consistent solution is indeed $E(\top, \bot, \top, \bot)$, and the set $\{\varphi \mid \Delta \models \Box\varphi\}$ contains:

$$DetRole(\text{“ A study in scarlet”}, Holmes) \wedge NovChar(\text{“ A study in scarlet”}, Holmes) \wedge$$
$$\wedge NovChar(\text{“ A life In Japan”}, Holmes) \wedge InJap(\text{“ A life In Japan”}, Holmes)$$

This implies the agent's query.

6 Discussion and Comparison with other approaches

As we have pointed out in the introduction, our view of context is that they defines the relativization of sentences. In this sense contexts are not predefined theories. The advantage of our approach is that it is easy to use because we can introduce contexts as parameters, names of a current situation. The computation of the solution to the system of equations is just a matter of substitutions. Actually it can be done directly within classical logic. The modal embedding ensures the existence of solutions and characterizes their form. In some sense the computation of a fixed point formula is analogous to the computation of a solution to a system of algebraic equations. A different view is that exploited by Lifschitz in [17], which is mainly concerned with varying the signature of the language from a context to another, by introducing keywords associated with predicate and function symbols. A context is thus identified by a first order theory and a subset of keywords, associated to the signature of the context. Lifschitz's notes are, thus, concentrated on mapping a context into another and on the relationship among different contexts. In this sense Lifschitz's approach is certainly crucial whenever one needs to merge knowledge bases using different languages or pass information from one to the another and it is faithful to the argument of McCarthy in [20], where it is required that assertions have to be transferred from a context to another. Also Guha's formalization [15] concentrates on first order properties of contexts, like the possibility of dealing with different interpretations of predicate symbols in different contexts. In our approach, clearly, we cannot manage a predicate symbol to have different arities, according to the context, since a context is a sentence that is computed by means of a system of equations. Our approach is useful in commonsense reasoning, whenever we have information only about the relativism of a set of sentences w.r.t. others, and thus contexts interpret such a relativism and the different fixed point solutions allows one to transcend the current context.

Lifschitz approach, on the other hand, is more useful, in commonsense reasoning, whenever information on contexts is given, that is, they are predefined theories and one needs to combine them or to exchange information from one to another.

In [4] a context is a label, and their approach exploit a multimodal logic. In some sense their view is that each context is an agent, the label of the modality. In this sense their point of view is close to our, since contexts cannot be viewed, like in Lifschitz approach as theories with a local signature. Now, the idea of [4], and similarly that of [10], allows to represent communications between contexts, which is another important aspect of formalizing contexts.

Also in the Situation Calculus [24] a situation can be seen as a context and progressing from a situation to another [18] is a way of passing information of actions from one context to another.

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