Point-Sensitive Circumscription

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Abstract

A decade ago, Pointwise Circumscription was proposed as a tool for formalizing common sense. In this paper, we revisit some of its uses and examine some cases in which it does not yield the intended behavior. Specifically, we explore the cases in which unsatisfiability may result from the presence of multiple minimal models (for McCarthy's Circumscription). Then, we present a form of circumscription, called *Point-Sensitive Circumscription*, that is a generalization of McCarthy's Circumscription and Lifschitz's Generalized Pointwise Circumscription. We illustrate how Point-Sensitive Circumscription handles these cases without losing the control over selective fine-grained variance of predicates and functions. Last, we compare the two tools and their potential uses in formalizing Theories of Action.

1 Introduction

Pointwise Circumscription, devised by Lifschitz in [Lif87] and [Lif89], is a nonmonotonic method defined along the intuitions of Circumscription [McC80],[McC86] with superior control of the minimization process. Since its debut, it has been used in formalizing Common Sense problems and solutions, such as in [DL94], formalizing some Entailment Classes in the theory of *Features and Fluents* (cf [San94]), and in [Ami97], using Pointwise Circumscription to formalize the system of [LR94]. It was argued in [Lif87] and [DL94] that Pointwise Circumscription has the power to be a tool for formalizing solutions for the Frame Problem.

In this paper, we show that there are cases in which Pointwise Circumscription requires the intended model to be a minimum, rather than merely minimal. We present a form of circumscription, called *Point-Sensitive Circumscription*, that is a generalization of McCarthy's Circumscription and Lifschitz's Generalized Pointwise Circumscription. Point-Sensitive Circumscription allows us to get a consistent theory where Pointwise Circumscription fails to do so, while maintaining similar control over the minimization process. For Point-Sensitive Circumscription, we allow a varied range of point-wiseness (the removal of elements of the minimized predicate separately), but maintain the same control over selective fine-grained variance of predicates and functions. We give semantics for our approach and demonstrate the different behavior using examples from the literature.

The paper is organized as follows. Sections 2 and 3 describe Pointwise Circumscription and its semantics and give an example that yields unsatisfiability. Sections 4 and 5 describe Point-Sensitive Circumscription and its semantics and reexamine the same example, this time with the new tool. Section 6 closely examines one commonsense application for which Pointwise Circumscription was used and compares the various options for that application.

For the rest of the paper, we use the following two conventions: (1) the relation $p < q$ between two propositions refers to the usual ordering of Boolean values ($false < true$), with $p \leq q$ allowing $p = q$; and (2) the relation $P < Q$ between two predicates with same arity refers to the strict subset relation (regarding P, Q as the relative sets of elements).

For background material on Circumscription and Pointwise Circumscription, the reader is referred to [Lif93] and [Lif87].

2 Pointwise Circumscription

2.1 The Formula

Pointwise Circumscription was first proposed in [Lif86] and then expanded in [Lif87] and [Lif89]. In [DL94] it was demonstrated that Pointwise Circumscription can be used to formalize various logics of action[1] in the framework of Features and Fluents [San94]. These logics include OCM (Original Chronological Minimization), PCM (Prototypical Chronological Minimization), PCMF (PCM with Filtering), CMON (Chronological Minimization of Occlusion with Nochange Premises) and CMOC (Chronological Minimization of Occlusion and Change).

[1] Some of the formalizations needed *filtering*.

The general idea behind Pointwise Circumscription is to capture the minimization idea of *Circumscription* [McC80], [McC86], while making the circumscription process more *distributed* (i.e., making it on a different subset of the theory/domain each time).

McCarthy's Circumscription formula

$$Circ[A(P,Z);P;Z] = A(P,Z) \wedge \forall p,z\ (A(p,z) \implies \neg(p < P))$$

says that in the theory A, with parameter relations and function sequences P, Z, P is a minimal element such that $A(P,Z)$ is still consistent, when we are allowed to vary Z in order to allow P to become smaller.

The basic case for Pointwise Circumscription is the formula

$$A(P) \wedge \forall x \neg [Px \wedge A(\lambda y(Py \wedge x \neq y))]$$

(where we minimize the subscript P again, varying nothing), intuitively saying that we remove one element at a time. One of the benefits of such an approach is the first-orderedness of the circumscription formula. This property disappears, though, in the most general form, which we use below.

Let $A(S_1, ..., S_n)$ be a sentence in which each S_i is a predicate symbol or a function symbol (in particular, it can be a 0-ary function symbol, i.e., an object constant). We want to minimize one of the predicate symbols from this list, S_{i_0} (Thus, S_{i_0} corresponds to P, and the other members of the list correspond to Z in the notation used above). Let us write $EQ_V(P,Q)$ ("P and Q are equal outside V") for

$$EQ_V(P,Q) \stackrel{def}{=} \forall x(\neg Vx \implies (Px \equiv Qx)) \tag{1}$$

for P, Q predicates or functions, and V a predicate, all with same arity. The *Pointwise circumscription* of S_{i_0} in A with S_i allowed to vary on V_i is, by definition,

$$A(S) \wedge \forall xs \neg [S_{i_0}x \wedge \neg s_{i_0}x \wedge \bigwedge_{i=1}^{n} EQ_{V_i x}(s_i, S_i) \wedge A(s)] \tag{2}$$

Here, S stands for $S_1, ..., S_n$, s is a list $s_1, ..., s_n$ of predicate and function variables corresponding to the predicate and function constants S, and $\lambda xu V_i(x,u)$ $(i = 1, ..., n)$ is a predicate without parameters that does not contain $S_1, ..., S_n$ and whose arity is the arity of S_{i_0} plus the arity of S_i. We denote (2) by $C_{PW}[A; S_{i_0}; S_1/V_1, ..., S_n/V_n]$.

2.1.1 An Intuitive Explanation of Pointwise Circumscription

The Pointwise Circumscription formula (2) has the property that we remove x from S_{i_0} while varying S (recall that S is the list of predicates/functions $S_1, ..., S_n$) in each of the $V_i x$ separately, fixing all the other parts of the domain, at that time.

In other words, the bracketed part of the formula says that, in the process of removing x from S_{i_0}, we allow only some parts of S to change. These parts are determined by $V_i x$ by saying that only where $V_i x$ is TRUE s_i may be different from S_i (recall that $EQ_{V_i x}(s_i, S_i)$ means that s_i is equal to S_i outside $\lambda u(V_i(x, u))$).

2.2 Semantics for Pointwise Circumscription

In [Lif87], Pointwise Circumscription was given semantic interpretation.

Take, without loss of generality, $i_0 = 1$. For a model $\mathcal{M}$ of $A(S)$, let $|\mathcal{M}|$ be the associated universe, and for every term, function or predicate a, $a^{\mathcal{M}}$ is the realization of a in $\mathcal{M}$.

Definition 2.1 *Let $\mathcal{M}_1, \mathcal{M}_2$ have the same universe U, and let $\xi \in U^k$, where k is the arity of S_1. We say that $\mathcal{M}_1 \leq^{\xi} \mathcal{M}_2$ iff*

1. *$K^{\mathcal{M}_1} = K^{\mathcal{M}_2}$ for every function or predicate constant K that is not in S,*
2. *for any $i = 1, ..., n$, $S_i^{\mathcal{M}_1}$ and $S_i^{\mathcal{M}_2}$ coincide on $\{\eta \mid \neg V_i^{\mathcal{M}_1}(\xi, \eta)\}$*
3. *$S_1^{\mathcal{M}_1}(\xi) \implies S_1^{\mathcal{M}_2}(\xi)$.*

Proposition 2.2 ([Lif87]) *Let $\mathcal{M} \in [\![A(S)]\!]$.*

$$\begin{aligned}\mathcal{M} \models C_{PW}[A; S_1; S_1/V_1, ..., S_n/V_n] \iff \\ \forall \mathcal{M}' \in [\![A(S)]\!]\ \forall \xi \in |\mathcal{M}|^k\ \neg(\mathcal{M}' \leq^{\xi} \mathcal{M} \wedge \mathcal{M} \not\leq^{\xi} \mathcal{M}')\end{aligned}$$

Other words, $\mathcal{M}$ is a model of $C_{PW}[A; S_1; S_1/V_1, ..., S_n/V_n]$ iff for each $\xi \in |\mathcal{M}|^k$, $\mathcal{M}$ is minimal relative to $\leq^{\xi}$.

3 A Counter-intuitive Example

In this section, we discuss only one simple example.

Let $A(P) = (P(X) \vee P(Y)) \wedge X \neq Y$. For this simple language (the only predicate is P, and there are exactly two object constant symbols X, Y), we get the Pointwise Circumscription formula

$$A(P) \wedge \forall xp \neg[Px \wedge \neg px \wedge EQ_{V_P x}(p, P) \wedge A(p)].$$

Recall that V_P limits the differences between p and P, according to the point x. Assume that for all x, $(V_P x) \equiv True$, so that p is allowed to differ from P with no limitations. We get the further simplified Pointwise Circumscription formula

$$A(P) \wedge \forall xp \neg[Px \wedge \neg px \wedge A(p)].$$

For $A(P) = (P(X) \vee P(Y)) \wedge X \neq Y$, we get the following result:

Proposition 3.1 *$C_{PW}[A(P); P; P/\lambda x.True] \models FALSE$.*

Proof

$$\begin{array}{l}
C_{PW}[A(P); P; P/\lambda x.True] \equiv \\
A(P) \wedge \forall xp \neg[P(x) \wedge \neg p(x) \wedge A(p)] \equiv \\
A(P) \wedge \forall xp \neg[P(x) \wedge \neg p(x) \wedge (p(X) \vee p(Y)) \wedge X \neq Y] \implies \\
A(P) \wedge \forall p \neg \left[\begin{array}{l} P(X) \wedge \neg p(X) \wedge (p(X) \vee p(Y)) \wedge X \neq Y \vee \\ P(Y) \wedge \neg p(Y) \wedge (p(X) \vee p(Y)) \wedge X \neq Y \end{array} \right] \equiv \\
\qquad\qquad A(P) \wedge \forall p \neg \left[\begin{array}{l} P(X) \wedge \neg p(X) \wedge p(Y) \wedge X \neq Y \vee \\ P(Y) \wedge \neg p(Y) \wedge p(X) \wedge X \neq Y \end{array} \right]
\end{array} \tag{3}$$

Let $p_x = \{X\}, p_y = \{Y\}$ be two possible values of p. Then, from the formula on the last line, we get

$$\begin{array}{l}
A(P) \wedge \forall p \neg \left[\begin{array}{l} P(X) \wedge \neg p(X) \wedge p(Y) \wedge X \neq Y \vee \\ P(Y) \wedge \neg p(Y) \wedge p(X) \wedge X \neq Y \end{array} \right] \implies \\
A(P) \wedge \neg \left[\begin{array}{l} P(X) \wedge \neg p_x(X) \wedge p_x(Y) \wedge X \neq Y \vee \\ P(Y) \wedge \neg p_x(Y) \wedge p_x(X) \wedge X \neq Y \vee \\ P(X) \wedge \neg p_y(X) \wedge p_y(Y) \wedge X \neq Y \vee \\ P(Y) \wedge \neg p_y(Y) \wedge p_y(X) \wedge X \neq Y \end{array} \right] \equiv \\
A(P) \wedge \neg \left[\begin{array}{l} FALSE \vee \\ P(Y) \wedge (TRUE \iff X \neq Y) \wedge X \neq Y \vee \\ P(X) \wedge (TRUE \iff X \neq Y) \wedge X \neq Y \vee \\ FALSE \end{array} \right] \equiv \\
A(P) \wedge \neg[(P(Y) \vee P(X)) \wedge X \neq Y] \equiv \\
P(X) \vee P(Y) \wedge X \neq Y \wedge \neg[(P(Y) \vee P(X)) \wedge X \neq Y] \equiv FALSE
\end{array}$$

■

This proposition reveals a limitation of Pointwise Circumscription, i.e., the requirement, in some cases, that P be a *minimum* rather than *minimal*[2].

This conclusion is supported by the semantics given by Lifschitz (proposition 2.2 above). Let $U = \{x, y\}$ be the set of elements in the universe. Let M_X, M_Y, M_{XY}, $M_\emptyset$ be the models with universe U, with X, Y interpreted to x, y, respectively, and the following interpretations for the predicate P:

- $P^{M_X} = \{X\}$.
- $P^{M_Y} = \{Y\}$.
- $P^{M_{XY}} = \{XY\}$.
- $P^{M_\emptyset} = \emptyset$.

Figure 1 below displays the different models for $A(P)$.

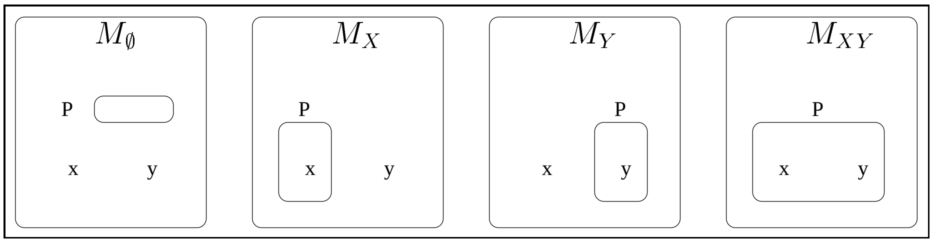

Figure 1: The four models of $A(P)$ with universe $U = \{x, y\}$.

There are exactly two elements in U, and thus there are two partial orders $\leq^x$,$\leq^y$ between the models. In order to check these partial orders, we first notice that the first two conditions of definition 2.1 are met by any pair of models from the four examined above. Using the third condition, we get the obvious

- $M_\emptyset \leq^y M_X \leq^y M_{XY}$ and $M_{XY} \not\leq^y M_X$.
- $M_\emptyset \leq^x M_Y \leq^x M_{XY}$ and $M_{XY} \not\leq^x M_Y$.

but also the somewhat less obvious

- $M_\emptyset \leq^x M_Y \leq^x M_X$ and $M_X \not\leq^x M_Y$.
- $M_\emptyset \leq^y M_X \leq^y M_Y$ and $M_Y \not\leq^y M_X$.

[2]Here *minimum* refers to an object that is smaller (in terms of the given partial order relation) than all other objects, while *minimal* refers to objects for which there is no smaller object.

Examining the conditions of proposition 2.2 for each one of the models, we reveal the following:

- $M_\emptyset$ is not a model of $C = C_{PW}[A(P); P; P/V_P]$ because it is not a model of $A(P)$.
- M_X is not a model of C because $M_Y \models A(P)$ and $M_Y \leq^x M_X$ and $M_X \not\leq^x M_Y$.
- M_Y is not a model of C because $M_X \models A(P)$ and $M_X \leq^y M_Y$ and $M_Y \not\leq^y M_X$.
- M_{XY} is not a model of C because $M_X \models A(P)$ and $M_X \leq^x M_{XY}$ and $M_{XY} \not\leq^y M_X$.

Thus, again, there is no model that satisfies C. Figure 2 below displays the two orders on models.

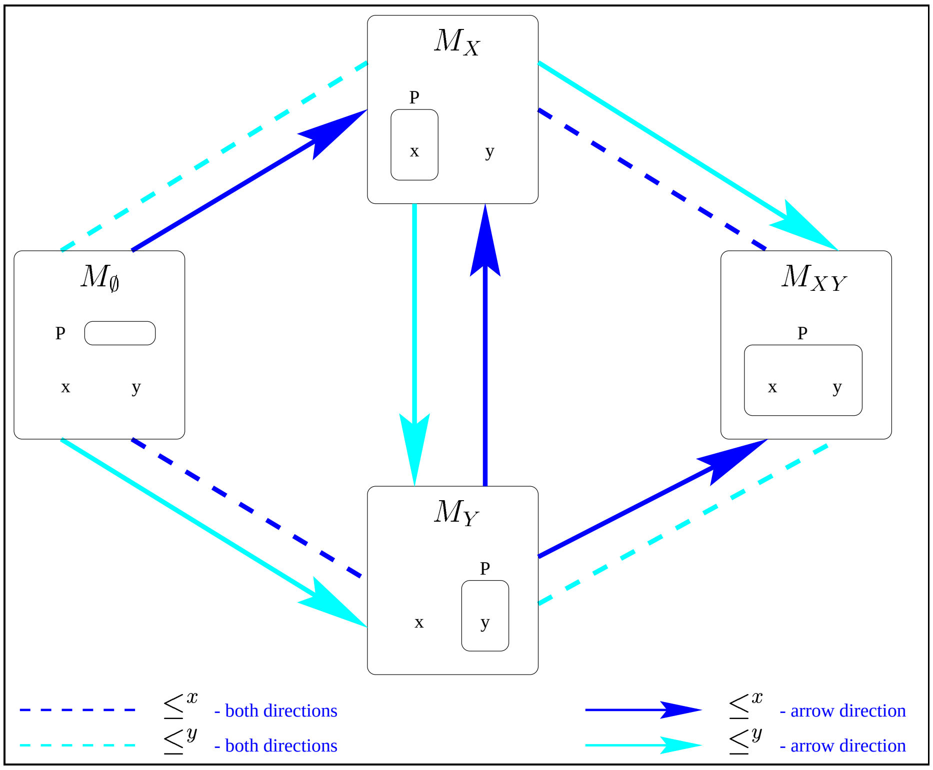

Figure 2: The orders $\leq^x$ and $\leq^y$. The different colors represent the different orders.

4 Point-Sensitive Circumscription

4.1 The formula

We present *Point-Sensitive Circumscription*, a modified version of Pointwise Circumscription in which the minimized predicate is minimized according to a *minimization domain*. This minimization domain may be a point and may be the complete set of elements. We preserve the ability to *select/vary* parts of the theory/domain dynamically.

It is important to notice at this point that Lifschitz already proposed a global circumscription that has some control over the varied domain (see formula (14) in [Lif87]). We shall compare that proposal with Pointwise Circumscription and Point-Sensitive Circumscription in section 6.

We use similar notations to those used in section 2. Let $A(S_1, ..., S_n)$ be a sentence in which each S_i is a predicate symbol or a function symbol (in particular, it can be a 0-ary function symbol, i.e., an object constant). We want to minimize one of the *predicate* symbols from this list, S_{i_0}.

In addition the definition of EQ_V in (1), let us write $LS_R(P, Q)$ ("P is smaller than Q in the region R", or $P \cap R \subsetneqq Q \cap R$) for

$$LS_R(P, Q) \stackrel{def}{=} \forall x(\neg Rx \implies (Px \implies Qx)) \wedge \exists x(Rx \wedge \neg Px \wedge Qx) \qquad (4)$$

for P, Q predicates or functions, and R a predicate, all with same arity.

The *Point-Sensitive Circumscription* of S_{i_0} in A with S_i allowed to vary on V_i, and S_{i_0} minimized using R, denoted $\mathcal{C}_{PS}[A; S_{i_0}/R; S_1/V_1, ..., S_n/V_n]$, is, by definition,

$$A(S) \wedge \forall xs \neg[LS_{Rx}(s_{i_0}, S_{i_0}) \wedge \bigwedge_{i=1}^{n} EQ_{V_i x}(s_i, S_i) \wedge A(s)] \qquad (5)$$

where S stands for $S_1, ..., S_n$, s is a list $s_1, ..., s_n$ of predicate and function variables corresponding to the predicate and function constants S, and $\lambda xuR(x, u)$, $\lambda xuV_i(x, u)$ $(i = 1, ..., n)$ are predicates without parameters that do not contain $S_1, ..., S_n$ and whose arity is the arity of S_{i_0} plus the arity of S_{i_0} and S_i, respectively (here $Rx = \lambda uR(x, u)$, $V_i x = \lambda uV(x, u)$).

4.1.1 An Intuitive Explanation

The bracketed part of the Point-Sensitive Circumscription formula (5) says that, in the process of making S_{i_0} smaller, we use R to define what "smaller" means and we allow only some parts of S to change. These parts are determined by Rx and $V_i x$ by saying that s_i may be different from S_i only where

V_ix is TRUE (recall that $EQ_{V_ix}(s_i, S_i)$ means that s_i is equal to S_i outside $\lambda u(V_i(x,u))$) and that we actually care if s_{i_0} is smaller only where Rx is TRUE.

Notice that the only difference between the Point-Sensitive Circumscription formula (5) and the Pointwise Circumscription formula (2) is in the first component. We changed $S_{i_0}x \wedge \neg s_{i_0}x$ (in (2)) to $LS_{Rx}(s_{i_0}, S_{i_0})$ (in (5)).

It is also important to notice that even for the cases where R is uniformly TRUE (in these cases, x is used only in the second component in the formula in parentheses), the second component is actually a conjunction of formulas, all using x. Therefore, although we do not require that x be removed from the predicate S_{i_0}, we do require that the variance of the different predicates/functions (including S_{i_0}) be correlated with x as an index.

4.2 Semantics

We follow the lines of [Lif87] with regard to the semantics for our Point-Sensitive Circumscription.

Take, without loss of generality, $i_0 = 1$. For a model $\mathcal{M}$ of $A(S)$, let $|\mathcal{M}|$ be the associated universe and, for every term, function or predicate a, $a^{\mathcal{M}}$ is the realization of a in $\mathcal{M}$. We write the corresponding definition to 2.1.

Definition 4.1 *Let $\mathcal{M}_1, \mathcal{M}_2$ have the same universe U, and let $\xi \in U^k$, where k is the arity of S_1. We say that $\mathcal{M}_1 \ll^{\xi} \mathcal{M}_2$ (a strict partial order) iff*

1. *$K^{\mathcal{M}_1} = K^{\mathcal{M}_2}$ for every function or predicate constant K that is not in S,*
2. *for any $i = 1, ..., n$, $S_i^{\mathcal{M}_1}$ and $S_i^{\mathcal{M}_2}$ coincide on $\{\eta \mid \neg V_i^{\mathcal{M}_1}(\xi, \eta)\}$*
3. *$LS_{R(\xi)}(S_1^{\mathcal{M}_1}, S_1^{\mathcal{M}_2})$.*

Let $[\![A(S)]\!]$ be the set of models of $A(S)$. The following proposition says that every model of the circumscription formula for $A(S)$ is minimal in $[\![A(S)]\!]$ according to all of the orders $\ll^{\xi}$, and vice versa.

Proposition 4.2 (Semantics for Point-Sensitive Circumscription) *Let $\mathcal{M} \in [\![A(S)]\!]$.*

$$\mathcal{M} \models C_{PS}[A; S_1/R; S_1/V_1, ..., S_n/V_n] \iff \\ \forall \mathcal{M}' \in [\![A(S)]\!]\ \forall \xi \in |\mathcal{M}|^k\ \neg(\mathcal{M}' \ll^{\xi} \mathcal{M})$$

PROOF For the forward direction, let $\mathcal{M} \models \mathcal{C}_{PS}[A; S_1/R; S_1/V_1, ..., S_n/V_n]$. Assume $\mathcal{M}' \in [\![A(S)]\!]$ $\mathcal{M}' \ll^{\xi} \mathcal{M}$ for some $\xi \in |\mathcal{M}|^k$. Then, $LS_{R(\xi)}(S_1^{\mathcal{M}'}, S_1^{\mathcal{M}})$, by definition 4.1 (requirement 3).

Take s to be $S^{\mathcal{M}'}$, the sequence of predicates/functions interpreting S (the sequence of predicate/function constant symbols) in $\mathcal{M}'$. For $x = \xi$ (ξ picked above), we get that in the model $\mathcal{M}$,

- $LS_{R(\xi)}(s_1(x), S_1(x))$, because $s = (s_1, ..., s_n)$ (that is our notation), and the way we picked ξ (s.t., $s_1 = S_1^{\mathcal{M}'}$, and $S_1^{\mathcal{M}'}(\xi) < S_1^{\mathcal{M}}(\xi)$), and
- $A(s)$, because $s = S^{\mathcal{M}'}$ and $M' \models A(S)$, and
- $\bigwedge_{i=1}^{n} EQ_{V_i x}(s_i, S_i)$, because of definition 4.1 (requirement 2).

This contradicts $\mathcal{M} \models \mathcal{C}_{PS}[A; S_1/R; S_1/V_1, ..., S_n/V_n]$, because we found x and s that satisfy the bracketed part of formula (5).

The reverse direction works by the same method. If $\mathcal{M}$ is $\ll^{\xi}$-minimal for every $\xi \in |\mathcal{M}|^k$, but does not satisfy formula (5), then there are s, x such that $LS_{R(\xi)}(s_1, S_1) \wedge \bigwedge_{i=1}^{n} EQ_{V_i x}(s_i, S_i) \wedge A(s)$. But then we can build a model $\mathcal{M}'$ with the same universe, taking all the constants other than those in S to be the same, and the constants of S take the values of s.

For $\mathcal{M}'$, we show now that $\mathcal{M}' \ll^{x} \mathcal{M}$. We follow the conditions of definition 4.1 one by one.

1. is satisfied by the construction of $\mathcal{M}'$.
2. is satisfied because s satisfies the second component of the bracketed part of the C_{PS} formula.
3. is satisfied because s satisfies the first component of the bracketed part of the C_{PS} formula.

We have found a model $\mathcal{M}'$ with $\mathcal{M}' \ll^{x} \mathcal{M}$. Contradiction. ∎

Lemma 4.3 *For $\xi \in |\mathcal{M}|^k$, $\ll^{\xi}$ is a strict partial order over models of $\mathcal{L}$.*

PROOF Irreflexivity is simply because $LS_R(P, Q)$ is irreflexive ($R \cap P \subsetneq Q \cap R$). It is now enough to prove transitivity. Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ with same domain, and $\xi \in |\mathcal{M}_1|^k$. Assume $\mathcal{M}_1 \leq^{\xi} \mathcal{M}_2$ and $\mathcal{M}_2 \leq^{\xi} \mathcal{M}_3$. We prove the conditions for $\mathcal{M}_1 \leq^{\xi} \mathcal{M}_3$ one by one.

1. For K not in S, $K^{\mathcal{M}_1} = K^{\mathcal{M}_2} = K^{\mathcal{M}_3}$.

2. For $i \in \{1, ..., n\}$, $S_i^{\mathcal{M}_1}, S_i^{\mathcal{M}_2}$ coincide on $\{\eta \mid \neg V_i^{\mathcal{M}_1}(\xi, \eta)\}$, and $S_i^{\mathcal{M}_2}, S_i^{\mathcal{M}_3}$ coincide on $\{\eta \mid \neg V_i^{\mathcal{M}_2}(\xi, \eta)\}$. Since $V_i^{\mathcal{M}_2} = V_i^{\mathcal{M}_1}$ (recall that $\lambda x u V_i(x, u)$ is a formula that contains no element from S), $S_i^{\mathcal{M}_2}, S_i^{\mathcal{M}_3}$ coincide on $\{\eta \mid \neg V_i^{\mathcal{M}_1}(\xi, \eta)\}$. Thus $S_i^{\mathcal{M}_1}, S_i^{\mathcal{M}_3}$ coincide on $\{\eta \mid \neg V_i^{\mathcal{M}_1}(\xi, \eta)\}$.

3. $LS_{R(\xi)}(S_1^{\mathcal{M}_1}, S_1^{\mathcal{M}_2})$ and $LS_{R(\xi)}(S_1^{\mathcal{M}_2}, S_1^{\mathcal{M}_3})$, and thus (transitivity of $\subsetneqq$) $LS_{R(\xi)}(S_1^{\mathcal{M}_1}, S_1^{\mathcal{M}_3})$.

Thus we proved all three requirements for $\mathcal{M}_1 \ll^{\xi} \mathcal{M}_3$ and transitivity is proved. Assymetry follows from transitivity and irreflexivity. ■

In the case of R uniformly true ($LS_{Rx}(P, Q) \equiv P \subsetneqq Q$), we get the following proposition.

Proposition 4.4 *Let A be a satisfiable theory with finitely many models, and let $Rx = Ry$ for all $x, y \in |\mathcal{M}|^k$. Then $\mathcal{C}_{PS}[A; S_1/R; S_1/V_1, ..., S_n/V_n]$ is satisfiable.*

Proof Since $[\![A]\!]$ is finite and $\ll^{\xi}$ is a partial order, there must be at least one minimal element $\mathcal{M}$ for $\ll^{\xi}$ from the models of A. If a minimal model $\mathcal{M}_1$ of $\ll^{\xi}$ is not a minimal element according to $\ll^{\zeta}$ then $\ll^{\zeta}$ has a smaller model $\mathcal{M}_2$. Since there are finitely many models, if we continue such a chain of models, we will eventually end up with a cycle. Without loss of generality assume that the cycle is $\mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_n, \mathcal{M}_1$. But then we get $LS_R(S_1^{\mathcal{M}_1}, S_1^{\mathcal{M}_n}), ...LS_R(S_1^{\mathcal{M}_2}, S_1^{\mathcal{M}_1})$, and thus $S_1^{\mathcal{M}_1} \subsetneqq S_1^{\mathcal{M}_n} \subsetneqq ... \subsetneqq ...S_1^{\mathcal{M}_2} \subsetneqq S_1^{\mathcal{M}_1}$, and thus $S_1^{\mathcal{M}_1} \subsetneqq S_1^{\mathcal{M}_1}$. Contradiction. ■

Last, we find out the following proposition.

Proposition 4.5 *1. Point-Sensitive Circumscription with $R \equiv TRUE$ and $V \equiv TRUE$ is equivalent to McCarthy's Circumscription.*

2. Point-Sensitive Circumscription with $Rx \equiv \lambda x'(x = x')$ is equivalent to Lifschitz's Generalized Pointwise Circumscription.

5 Examples using Point-Sensitive Circumscription

Let us first recall the example from section 3. Let $A(P) = (P(X) \vee P(Y)) \wedge X \neq Y$. Let $(V_P x) \equiv TRUE$ for all x, so that p is allowed to differ from P with no limitations. Let $R \equiv TRUE$, so that $LS_{Rx}(P, Q) \equiv (P \subsetneqq Q)$. We get the simplified Point-Sensitive Circumscription formula

$$A(P) \wedge \forall p \neg[p < P \wedge A(p)].$$

For $A(P) = (P(X) \vee P(Y)) \wedge X \neq Y$, we get the following result (contrary to proposition 3.1 for the Pointwise Circumscription case):

Proposition 5.1 $\mathcal{C}_{PS}[A(P); P/R; P/\lambda x.True] \equiv (\forall x(P(x) \iff x = X)) \vee (\forall x(P(x) \iff x = Y)) \wedge X \neq Y$.

Proof

$$\mathcal{C}_{PS}[A(P); P; P/\lambda x.True] \equiv A(P) \wedge \forall p \neg[p < P \wedge A(p)]$$

At this point, all we need to notice is that we got McCarthy's Circumscription which is known to have the required equivalence. ■

Let us now examine the situation as seen by the semantics of proposition 4.2. Let $U = \{x, y\}$ be the set of elements in the universe. We explore the same models discussed in section 3, M_X, M_Y, M_{XY} and $M_\emptyset$. Figure 1 above displays the different models for $A(P)$. Figure 2 above then describes the two orders $\leq^x$, $\leq^y$ between the models. Now we examine the semantics of proposition 4.2.

First, we notice that both $\ll^X, \ll^Y$ exactly correspond to the $\subsetneq$ relation for the respective P's. Therefore, we get

- $M_\emptyset \ll^x M_X \ll^x M_{XY}$ and $M_{XY} \not\ll^x M_X$ and $M_X \not\ll^x M_Y$.
- $M_\emptyset \ll^x M_Y \ll^x M_{XY}$ and $M_{XY} \not\ll^x M_Y$ and $M_Y \not\ll^x M_X$.

Examining the conditions of proposition 4.2 for each one of the models, we reveal the following:

- $M_\emptyset$ is not a model of $C = \mathcal{C}_{PS}[A(P); P; P/V_P]$ because it is not a model of $A(P)$.
- M_{XY} is not a model of C because $M_X \models A(P)$ and $M_X \ll M_{XY}$.
- M_X *is* a model of C because $M_X \models A(P)$ and $M_{XY} \not\ll M_X$ and $M_Y \not\ll M_X$.
- M_Y *is* a model of C because $M_Y \models A(P)$ and $M_{XY} \not\ll M_Y$ and $M_X \not\ll M_Y$.

Thus, again, there are exactly two models that satisfies C, one in which $P = \{X\}$ and the other in which $P = \{Y\}$. Figure 3 below displays the two orders on models.

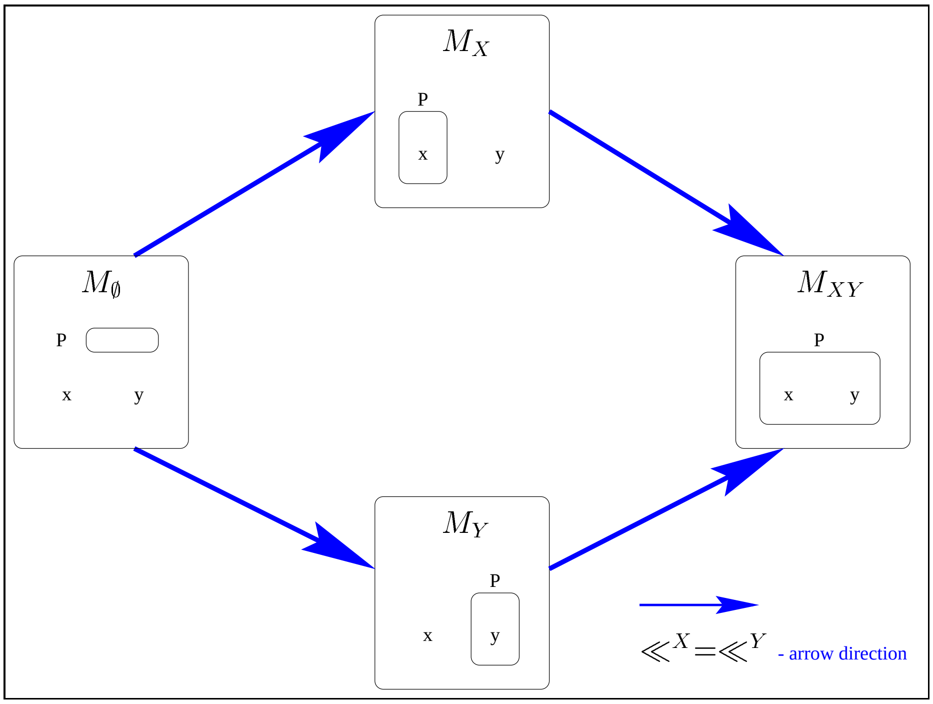

Figure 3: The orders $\ll^X$ and $\ll^Y$.

6 Applying Point-Sensitive to Theories of Action

We examine one application of Pointwise Circumscription for the formalization of Prototypical Chronological Minimization (PCM), which is one known solution to the Frame Problem (for some ontological classes[3]). Finally, we compare it with the proposed Point-Sensitive Circumscription and with Generalized Global Circumscription (as proposed in [Lif87] (formula (14))).

6.1 PCM Using Pointwise Circumscription

Doherty and Lukaszewicz [DL94] examined various solutions to the Frame Problem, implementing them using either McCarthy's Circumscription or Lifschitz's Generalized Pointwise Circumscription (formula 2 above). We briefly review some of the theory of Features and Fluents [San94] and one of the solutions that was examined in [DL94], namely, Prototypical Chronological Minimization (PCM), for which Pointwise Circumscription was used.

[3]See [San94].

6.1.1 The general theory for PCM

The language $\mathcal{L}(FL)$ is a sorted first-order language with equality. There are two domain independent sorts, $\mathcal{T}$ for time points (t, s are variables of that sort) and $\mathcal{F}$ for propositional fluents (f, g are variables of that sort), which are functions from time to truth values. We include the predicate symbols $Holds, Clip$ of type $\mathcal{T} \times \mathcal{F}$ and the predicate symbols $<, \leq$ of type $\mathcal{T} \times \mathcal{T}$ (total orders on time points).

Let $\Gamma_C = \Gamma_{OBS} \cup \Gamma_{SCD} \cup \Gamma_{UNA}$ denote the set of formulas in $\mathcal{L}(FL)$ corresponding to the given scenario description, where Γ_{OBS} are the observation axioms (e.g., $Holds(f, s)$), Γ_{SCD} are the schedule axioms (a form of "effect axioms") and Γ_{UNA} are the appropriate unique name axioms for the fluent constants. In addition, Γ_{PER} denotes the set containing the two persistence axioms

$$\begin{array}{l} \forall f, t. Holds(t, f) \Rightarrow [Holds(t+1, f) \bar{\vee} Clip(t+1, f)] \\ \forall f, t. \neg Holds(t, f) \Rightarrow [\neg Holds(t+1, f) \bar{\vee} Clip(t+1, f)]. \end{array}$$

For the logic of PCM, we include the following. Let $\Gamma(Clip, Holds)$ denote $\Gamma_C \cup \Gamma_{PER}$. Let $V(t, f) = \lambda t' f'.[\langle t, f \rangle = \langle t', f' \rangle \vee t < t']$ and $U(t, f) = \lambda t' f'.[t \leq t']$. Then, $C_{PW}[\Gamma(Clip, Holds); Clip; Clip/V(t, f); Holds/U(t, f)]$ is

$$\begin{array}{l} \Gamma(Clip, Holds) \wedge \\ \forall t, f. \forall clip, holds. \neg[Clip(t, f) \wedge \neg clip(t, f) \wedge \\ \qquad\qquad EQ_{V(t,f)}(Clip, clip) \wedge EQ_{U(t,f)}(Holds, holds) \wedge \\ \qquad\qquad \Gamma(clip, holds)] \end{array} \tag{6}$$

The intention is that when minimizing $Clip$ at time t for fluent f, the only part of $Clip$'s extension that will vary is the point $\langle t, f \rangle$ itself or $\langle t', f' \rangle$ for $t' > t$ and arbitrary f'. In addition, $Holds$ is varied only for time points $t' \geq t$.

6.1.2 YSP variant 1: global minimization

Let us examine a simplified variant of the Yale Shooting Scenario (YSS) ([HM86]). The world is characterized by two truth values, DEAD and ALIVE[4], describing the state of a Turkey. There is exactly one action, $Wait$, which causes nothing. Initially the Turkey is ALIVE. The course of happenings is simple: We simply Wait. The intended meaning is that the turkey stays alive as a result of Wait.

[4] In this simplification we assume that the YSS gun is LOADED all along.

The theory is as follows (describing observations (obs) and schedules (scd) for time segment of the form $[t1, t2]$):

$$\begin{array}{ll} obs1 & [0,\infty)ALIVE \leftrightarrow \neg DEAD \\ obs2 & [0]ALIVE \\ scd1 & 0 < \mathbf{t} \end{array} \tag{7}$$

The intended result of applying (6) is that the turkey is still alive at time $\mathbf{t}$, but this is not what we get. Applying (6) yields two minimal models:

- $obs3$ $[\mathbf{t}]ALIVE, \neg DEAD$.
- $obs3$ $[\mathbf{t}]\neg ALIVE, DEAD$.

The reason for the two minimal models is the insistance on minimizing one fluent at a time, while not allowing other changes for other fluents for the same time point.

6.1.3 YSP variant 2: nondeterministic actions

One proposed solution might be to minimize change for all the fluents at once for a given time point. For this solution, we set the policy V to be

$$V(t, f) = \lambda t' f'.[t \leq t']. \tag{8}$$

Trying to use (8), we encounter the problem discussed in section 3, i.e., that in the presence of fluents that are minimized in a circular fasion (i.e., minimizing P while varying Q and vice versa), the minimimal models must agree on the involved predicates.

Let us look at another modified version of the YSP. Assume that there are two turkeys, Turkey1 and Turkey2. As a result of the gun being shot, exactly one dies. For this, we have the propositions ALIVE1, DEAD1, ALIVE2, DEAD2, LOADED. Initially, both turkeys are alive, and the gun is LOADED. Here we have the sequence of waiting and then shooting.

$$\begin{array}{ll} obs1 & [0,\infty)(ALIVE1 \leftrightarrow \neg DEAD1) \\ obs2 & [0,\infty)(ALIVE2 \leftrightarrow \neg DEAD2) \\ obs3 & [0]ALIVE1, ALIVE2, LOADED \\ scd1 & [\mathbf{s}]LOADED \rightarrow [\mathbf{s},\mathbf{t}]ALIVE1 := TF \\ obs4 & [\mathbf{s}]LOADED \rightarrow [\mathbf{s},\mathbf{t}]ALIVE2 \leftrightarrow \neg ALIVE1 \\ scd2 & 0 < \mathbf{s} < \mathbf{t} \end{array} \tag{9}$$

$scd1$ says that, given that at time $\mathbf{s}$ the gun is loaded, then between $\mathbf{s}$ and $\mathbf{t}$, Turkey1 either dies or stays alive (as a result of the shooting action).

$obs4$ then says that the result of the action is that exactly one turkey dies (notice that we had to write $obs4$ as an observation because of the cues of the language). There are two minimal models for this scenario, one in which Turkey1 is eventually dead, and another in which Turkey2 is eventually dead. In this case, Pointwise circumscription will incur inconsistency.

6.2 Comparing Solutions

It is important to note that the problems we describe above do not affect the findings of [DL94], because these problems fall outside the ontological class that the formalization of [DL94] tried to formalize, namely the ontological class $\mathcal{K}p\text{-}IAex$. This class assumes that there are no observations outside the initial state. For the purposes of [DL94], a domain constraint such as $DEAD1 \leftrightarrow \neg ALIVE1$ is an observation for time points later than 0. Thus, to include this more *advanced* class, we need *filtering*[5].

We can do better than that using Point-Sensitive Circumscription. Let $R(t,f) = \lambda t' f'[t' = t]$. As a result of applying the circumscription policy (8) to theory (9), we get $\mathcal{C}_{PS}[\Gamma(Clip, Holds); Clip/R; Clip/V, Holds/U] =$

$$\begin{array}{l} \Gamma(Clip, Holds) \wedge \\ \forall t, f \forall clip, holds \neg [LS_{R(t,f)}(clip, Clip) \wedge \\ \qquad\qquad EQ_{V(t,f)}(Clip, clip) \wedge EQ_{U(t,f)}(Holds, holds) \wedge \\ \qquad\qquad \Gamma(clip, holds)] \end{array} \tag{10}$$

This formula minimizes $Clip$ for each time point for all fluents together (rather than per fluent). We get the following two minimal models

- $obs5$ $[\mathbf{t1}]\neg ALIVE1, DEAD1, ALIVE2, \neg DEAD2, LOADED.$
- $obs5$ $[\mathbf{t1}]ALIVE1, \neg DEAD1, \neg ALIVE2, DEAD2, LOADED.$

which is exactly the outcome we wanted.

We do not claim that Point-Sensitive Circumscription can formalize the entire $\mathcal{K}\text{-}IA$ ontological class, but that it can take you farther than Pointwise Circumscription can. Another example in which the semantics of [LR94] are formalized using Point-Sensitive Circumscription is detailed in [Ami97].

Last, let us examine another proposal made in [Lif87], namely, an expanded version of *Global Circumscription*:

$$Circ_G[A; P; Z/V] = A(P,Z) \wedge \forall pz \neg [EQ_V(z,Z) \wedge A(p,z) \wedge p < P]$$

[5] Notice that our example (9) is in $\mathcal{K}\text{-}IAenx$.

This is a reduced version of C_{PS} in which we can not use different domains V_x for different points in the minimization. In the example above, it is essential to vary the right domains at the right time (i.e., vary the fluents only for time points later than the current time point). Therefore, this proposal is not enough for the representation of the preference needed above. As a side issue, notice that by taking $R \equiv TRUE$ $V(t, f) \equiv V$, we get $C_{PS} \equiv Circ_G$.

7 Conclusions

In the Theories of Action context, Pointwise Circumscription is sufficient if we restrict ourselves to deterministic actions and to cases in which there are no domain constraints and no action's effects are dependent on other actions' effects.

We showed that Pointwise Circumscription does not yield the right conclusions for some cases in which there are multiple minimal models (for McCarthy's Circumscription). This is due to the "point-wise-ness" of that form of circumscription. We described Point-Sensitive Circumscription, a generalization of both McCarthy's Circumscription and Lifschitz's Pointwise Circumscription, and showed that it is useful for Theories of Action. Point-Sensitive Circumscription allows us to choose our degree of "point-wise-ness" (as suggested by the fact that both Circumscription and Pointwise Circumscription are its special cases) by controlling the domain used for the comparison of predicates. Since most uses of Pointwise Circumscription to date (e.g., [DL94]) refer mostly to the last property (variance control) rather than the first one ("point-wise-ness"), Point-Sensitive Circumscription is a useful substitute.

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