Circumscriptions from what they cannot do
(preliminary report)

Yves Moinard * and Raymond Rolland †

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Abstract

We show that, in the finite propositional case, traditional circumscriptions can be fully described only from the formulas which can never come as a result of the given circumscription (the "inaccessible formulas"). Some work has already been done on the subject. Siegel and Forget have introduced the general notion of X-logic, and they have considered the case of finite propositional circumscriptions. However, their result is restricted to the case where no varying proposition appears, which is known to be a severe restriction in terms of expressivity. We extend this result in the finite propositional case to any cumulative preferential entailment, thus in particular to any "traditional circumscription", that is we allow varying propositions. Moreover, we describe the smallest possible set which can be used for this purpose.

We exhibit a spectacular duality between the traditional approach of a given cumulative preferential entailment, and the approach through the inaccessible formulas of this preferential entailment, i.e. the X-logic approach.

We extend these results outside the framework of pure propositional preferential entailments.

*IRISA, Campus de Beaulieu, 35042 RENNES-Cedex FRANCE, tel.: (33) 2 99 84 73 13, e-mail: moinard@irisa.fr

†IRMAR, Campus de Beaulieu, 35042 RENNES-Cedex FRANCE, tel.: (33) 2 99 28 60 19, Raymond.Rolland@univ-rennes1.fr

1 Introduction

We show that, in the finite propositional case, cumulative circumscriptions can be fully described only from the formulas which can never come as a result of the given circumscription (the "inaccessible formulas"). Some work has already been done on the subject. [Suc87, Suc93] have given nice results in the predicate calculus case, but only for circumscriptions without varying predicate, which is a rather severe restriction, as known till the begining of circumscription (see e.g. [McC86, PM86]). Siegel and Forget have introduced the general notion of X-logic in [SF96], and they have considered the case of finite propositional circumscriptions, again with the restriction that no varying proposition appears. We extend the results in the finite propositional case to any cumulative preferential entailment, thus in particular to any "traditional circumscription", that is we allow varying propositions.

We exhibit a spectacular duality between the traditional approach of a given cumulative preferential entailment, and the approach through its inaccessible formulas, i.e. the X-logic approach. We use the smallest possible set of inaccessible formulas which suffices to reconstruct the preferential entailment. We do not know whether our results could be extended to "real" (i.e predicate) circumscriptions, as this is not immediate and is planned for future work. If so, this would extend Suchenek's results when there are varying predicates.

In order to fix the definitions, we introduce also the notion of S-mapping, which plays a crucial role in many non-monotonic reasoning methods, such as default reasoning.

In section 2, we describe our technical framework (propositional logic) and the notions of X-mappings and S-mappings. In terms of [KLM90], S-mappings are inference relations satisfying LLE and RW and some strong reflexivity. We give the properties of X-mappings necessary for us. We show why the inaccessible formulas play a central role. In section 3, we examine the particular X-mappings which are pre-circumscriptions (in terms of [KLM90], inference relations satisfying LLE, RW, AND and reflexivity), making precise and extending various results in [SF96]. In section 4, we give some properties of propositional preferential entailments and of a notion originating in [KLM90], called here multi preferential entailment. In section 5 we give, for the finite case, a necessary and sufficient condition for which an X-mapping is a preferential entailment, and also a necessary and sufficient condition for which a preferential entailment is an X-mapping. We exhibit the smallest possible set X, and a spectacular duality between this set structured by $\models$ and the set of complete theories structured by the converse of the relation defining the preferential entailment. Using multi preferential entailments, we extend these results to any pre-circumscriptions, instead of just preferential entailments. In section 6, we apply our results to traditional circumscriptions, still in the finite case.

2 Notations and first definitions

We work in a classical logic, and in fact in a propositional logic. Thus, our logic is compact (and complete: $\models=\vdash$). Also, when dealing with preferential entailments, a relation between models may be assimilated to a relation between complete theories.

$\boldsymbol{\mathcal{L}}$ is the name of the logic considered, which contains some propositional symbols noted $A, B, C, \cdots$. $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is the set (at most enumerable) of all the propositional symbols of $\boldsymbol{\mathcal{L}}$. As usual, $\boldsymbol{\mathcal{L}}$ will denote either the logic (its vocabulary and rules) or the set of all the formulas existing in this logic.

$\mathcal{F}$ is the set of all the sets of formulas in $\boldsymbol{\mathcal{L}}$. The set of *theories* is $\mathcal{T} = \{\mathcal{T}/\mathcal{T} \in \mathcal{F},\ \mathcal{T} = Th(\mathcal{T})\}$ where $Th(\mathcal{T}) = \{\varphi/\mathcal{T} \models \varphi\}$. Letters $\varphi, \psi, \cdots$ will denote formulas in $\boldsymbol{\mathcal{L}}$. Thus, "for any φ" will mean "for any formula φ in $\boldsymbol{\mathcal{L}}$". Letters $X, Y, \mathcal{S}, \mathcal{T}, \cdots$ will denote elements of $\mathcal{F}$ (often, $\mathcal{T}$ will be used for elements in $\mathcal{T}$, but then this will be precised each time).

$\perp$ denotes the false formula, and $\top$ the true formula. $\mathcal{C}$ denotes the set of all the *complete* theories $\mathcal{T}$ in $\boldsymbol{\mathcal{L}}$: $\varphi \in \mathcal{T}$ exclusive or $\neg\varphi \in \mathcal{T}$ for any φ.

We use the notations $\mathcal{T}_1 \sqcup \mathcal{T}_2 = Th(\mathcal{T}_1 \cup \mathcal{T}_2)$, $\bigsqcup_{i\in I} \mathcal{T}_i = Th(\bigcup_{i\in I} \mathcal{T}_i)$ and, abusively, $\mathcal{T} \sqcup \varphi$ for $\mathcal{T} \sqcup \{\varphi\}$.

For any $\mathcal{T} \in \mathcal{T}$, $\Gamma(\mathcal{T})$ denotes the set of all the complete theories which entail (i.e. contain) $\mathcal{T}$. We know that $\mathcal{T} = \bigcap_{\mathcal{T}'' \in \Gamma(\mathcal{T})} \mathcal{T}''$.

For any $X \in \mathcal{F}$, $\neg(X)$ denotes the set $\{\neg\varphi \ / \ \varphi \in X\}$.

We will in fact consider the quotient of the notion of formula by logical equivalence, thus $A \vee \neg A = B \vee \neg B = \top$ and, if $\mathcal{V}(\boldsymbol{\mathcal{L}})$ has n elements, $\mathcal{C}$ is a set of 2^n elements, $\mathcal{T}$ (and $\boldsymbol{\mathcal{L}}$) has 2^{2^n} elements.

Definition 2.1 [SF96] For any $X \in \mathcal{F}$, we define the mapping f_X from $\mathcal{F}$ to $\mathcal{F}$ by: $\varphi \in f_X(\mathcal{T})$ iff $(\mathcal{T} \sqcup \varphi) \cap X \subseteq Th(\mathcal{T})$.

Example 2.2 $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \{A, B\}$, X is the singleton $\{A \wedge B\}$. We have 16 elements in $\mathcal{T}$, thus we may list exhaustively all the $f_X(\varphi)$:
$f_X(\perp) = f_X(A \wedge B) = \boldsymbol{\mathcal{L}} = Th(\perp)$,
$f_X(\varphi) = Th(\varphi)$ for $\varphi \in \{A \wedge \neg B, \neg A \wedge B, \neg A \wedge \neg B\}$,

$f_X(A) = Th(A \wedge \neg B), f_X(B) = Th(\neg A \wedge B)$, $f_X(A \Leftrightarrow B) = Th(\neg A \wedge \neg B)$, $f_X(A \not\Leftrightarrow B) = Th(\neg A \wedge B) \cup Th(A \wedge \neg B)$,
$f_X(\neg B) = f_X(A \vee \neg B) = Th(A \wedge \neg B) \cup Th(\neg A \wedge \neg B)$,
$f_X(\neg A) = f_X(\neg A \vee B) = Th(\neg A \wedge B) \cup Th(\neg A \wedge \neg B)$,
$f_X(A \vee B) = Th(A \wedge \neg B) \cup Th(\neg A \wedge B)$,
$f_X(\neg A \vee \neg B) = f_X(\top) = Th(A \wedge \neg B) \cup Th(\neg A \wedge B) \cup Th(\neg A \wedge \neg B)$.

$f_X(\varphi)$ is not necessarily in $\mathcal{J}$. We will see below that when X is a singleton, $f_X(\varphi)$ is either $\boldsymbol{\mathcal{L}}$ or a union of complete theories. $\square$

Here are useful immediate characterisations of $f_X(\mathcal{T})$:

Property 2.3 For any $\mathcal{T} \in \mathcal{F}$, $\varphi \in f_X(\mathcal{T})$ iff
1) For any $x \in X - Th(\mathcal{T})$, $x \notin \mathcal{T} \sqcup \varphi$, iff
2) For any $x \in X - Th(\mathcal{T})$, $\neg\varphi \notin \mathcal{T} \sqcup \neg x$.

From definition 2.1, the three following properties are obvious:,

Property 2.4 [SF96] f_X satisfies: 1) Reflexivity (Ref): $\mathcal{T} \subseteq f_X(\mathcal{T})$.
2) (LLE): if $Th(\mathcal{T}_1) = Th(\mathcal{T}_2)$, then $f_X(\mathcal{T}_1) = f_X(\mathcal{T}_2)$.
3) Right Weakening (RW1): if $\varphi \in f_X(\mathcal{T})$ and $\varphi \models \psi$, then $\psi \in f_X(\mathcal{T})$.

LLE, or Left Logical Equivalence (see e.g. [KLM90]), means that f_X is in fact a mapping from $\mathcal{J}$ to $\mathcal{F}$. (RW1) means that f_X is a mapping in some subset of $\mathcal{F}$ that we will examine now.

Definitions 2.5 • $\boldsymbol{\mathcal{S}}$ is the set of all the *S-sets*, the subsets of $\boldsymbol{\mathcal{L}}$ which are *unary closed*: $\mathcal{S} \in \boldsymbol{\mathcal{S}}$ iff $\mathcal{S} = \{\varphi/$ there exists $\psi \in \mathcal{S}, \psi \models \varphi\}$.

•The $\wedge$-*closure* of $\mathcal{T} = \{\varphi_i\}_{i \in I}$ is the set $\mathcal{T}^\wedge = \{\bigwedge_{j \in J} \varphi_j /$ for any finite $J \subseteq I\}$. The $\vee$-closure $\mathcal{T}^\vee$ is defined similarly. $\top \in \mathcal{T}^\wedge, \perp \in \mathcal{T}^\vee$ $(J = \emptyset)$.

S-sets appear in many places in the literature about formalization of common sense reasoning: the set of default extensions is in $\boldsymbol{\mathcal{S}}$, as are all the objects which "satisfy right weakening".

Remind that, as our logic is compact, we get (Prop$\sqcup - \subseteq$):
if $\mathcal{T}_i \subseteq \mathcal{T}_{i+1}$, $\mathcal{T}_i \in \mathcal{J}$ for any $i \in \mathbb{N}$, then $\bigcup_{i \in \mathbb{N}} \mathcal{T}_i = \bigsqcup_{i \in \mathbb{N}} \mathcal{T}_i$.

Property 2.6 Let $\mathcal{S}$ be an S-set. Then $\mathcal{S} = \bigcup_{i \in I} \mathcal{T}_i$ where for any $i \neq j \in I$ we have $\mathcal{T}_i \in \mathcal{J}$, $\mathcal{T}_i \not\subseteq \mathcal{T}_j$ and $\mathcal{T}_j \not\subseteq \mathcal{T}_i$.

Moreover, the set $\{\mathcal{T}_i\}_{i \in I}$ having this property is unique. We call this the *canonical writing* of $\mathcal{S}$.

Proof: We define the set $\mathcal{J}_{\mathcal{S}} = \{\mathcal{T}_j\}_{j \in J}$ of all the elements $\mathcal{T}_j \in \mathcal{J}$ such that $\mathcal{T}_j \subseteq \mathcal{S}$. We get $\mathcal{S} = \bigcup_{j \in J} \mathcal{T}_j$. We define $I = \{i \in J \ / \ \mathcal{T}_i$ is maximal for $\subseteq$ in the set $\mathcal{J}_{\mathcal{S}}\}$. This set is well defined from (Prop$\sqcup - \subseteq$) and uniqueness is obvious. $\square$

Definition 2.7 The *kernel* of an S-set is the set $K(\mathcal{S})$ which satisfies one of the two equivalent conditions:

1. $\mathcal{S} = \bigcup_{i \in I} \mathcal{T}_i$ being the canonical writing of $\mathcal{S}$, $K(\mathcal{S}) = \bigcap_{i \in I} \mathcal{T}_i$.
2. a) $K(\mathcal{S}) \in \mathcal{J}$ and $K(\mathcal{S}) \subseteq \mathcal{S}$,
   b) For any $\mathcal{T} \in \mathcal{J}, \mathcal{T} \subseteq \mathcal{S}$, there exists some $\mathcal{T}' \in \mathcal{J},\ \mathcal{T} \subseteq \mathcal{T}' \subseteq \mathcal{S}$ such that $K(\mathcal{S}) \subseteq \mathcal{T}'$.
   c) $K(\mathcal{S})$ is maximal for $\subseteq$ respecting a) and b).

The proof of the equivalence is easy, thus we omit it here.

From the definitions of $\mathfrak{S}$ and $K(\mathcal{S})$, we get immediately:

Property 2.8 • For any $\mathcal{S} \in \mathfrak{S}$, we have $Th(\top) \subseteq K(\mathcal{S}) \subseteq \mathcal{S}$.
• An S-set $\mathcal{S}$ is in $\mathcal{J}$ iff $\mathcal{S} = K(\mathcal{S})$.

From now on, we consider the f_X as defined in definition 2.1 as a mapping from $\mathcal{J}$ to $\mathfrak{S}$, as this seems for us the good way to comprehend the notion of f_X. Still, but this is only a notational convenience, we will use the abbreviation $f(\mathcal{T})$ for $f(Th(\mathcal{T}))$ if $\mathcal{T} \notin \mathcal{J}$.

We have more than reflexivity as seen in property 2.4.

Property 2.9 • For any set X, f_X has the property of *strong Reflexivity* (sRef): For any $\mathcal{T} \in \mathcal{J}$ we have $\mathcal{T} \subseteq K(f_X(\mathcal{T}))$.
• For any set X, f_X has the property of *restricted And* (rAND): If $\mathcal{T} \in \mathcal{J}, \mathcal{T}_1 \in \mathcal{J}, \mathcal{T}_1 \subseteq f_X(\mathcal{T})$ then $\mathcal{T} \sqcup \mathcal{T}_1 \subseteq f_X(\mathcal{T})$.

Thus, f_X has the property of *restricted And (formula version)* (rAND1): If $\psi \in Th(\mathcal{T})$ and $\varphi \in f_X(\mathcal{T})$, then $\varphi \wedge \psi \in f_X(\mathcal{T})$.

Proof: It is easier to prove (rAND) first. This property is obvious once we remark that for any $\mathcal{T}' \in \mathcal{J}$, $\mathcal{T}' \subseteq f_X(\mathcal{T})$ iff $(\mathcal{T}' \sqcup \mathcal{T}) \cap X \subseteq \mathcal{T}$. (sRef) comes easily from (rAND), using the canonical writing of $K(f_X(\mathcal{T}))$.

The converse of (rAND) is obviously satisfied by any X-mapping also. It is a consequence of a trivial property, which we call (RW): If $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$ are in $\mathcal{J}, \mathcal{T}_1 \sqcup \mathcal{T}_2 \subseteq f_X(\mathcal{T})$ then $\mathcal{T}_1 \subseteq f_X(\mathcal{T})$.

As for any property, the full version implies the formula version: (rAND) implies (rAND1), and (RW) implies (RW1). $\square$

Now, we know that f_X satisfies (LLE), (RW) and (sRef). This motivates the following definitions:

Definitions 2.10 • An S-mapping is any application f from $\mathcal{J}$ to $\mathcal{S}$ which satisfies $\mathcal{T} \subseteq K(f(\mathcal{T}))$ for any $\mathcal{T} \in \mathcal{J}$.

• An S-mapping satisfies *(AND)* iff: For any $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$, if $\mathcal{T}_1 \cup \mathcal{T}_2 \subseteq f(\mathcal{T})$ then $\mathcal{T}_1 \sqcup \mathcal{T}_2 \subseteq f(\mathcal{T})$. As the logic is compact, this is equivalent to the formula version *(AND1)*: If $\varphi \in f(\mathcal{T})$ and $\psi \in f(\mathcal{T})$ then $\varphi \wedge \psi \in f(\mathcal{T})$.

• A *pre-circumscription*, is a mapping f from $\mathcal{J}$ to $\mathcal{J}$ where $\mathcal{T} \subseteq f(\mathcal{T})$.

• When $\mathcal{T}$ is in $\mathcal{F} - \mathcal{J}$, we use the abbreviation $f(\mathcal{T}) = f(Th(\mathcal{T}))$.

• An *X-mapping* is an S-mapping f which is equal to some f_X.

The notion of S-mapping extends the more traditional notion called "pre-circumscription" in [MR94a, MR94b], notion used, explicitely or not, in many texts about non monotonic reasoning, e.g. in texts about preferential entailments. Notice that default logic [Rei80] may be naturally expressed in terms of S-mappings. An S-mapping f is a pre-circumscription iff $f(\mathcal{T}) \in \mathcal{J}$ for any theory $\mathcal{T}$, i.e. iff it satisfies (AND), i.e iff it satisfies (AND1).

Any X-mapping f_X is an S-mapping, as it satisfies (LLE), (RW) and (sRef). Any S-mapping satisfies also (rAND) and even the stronger version (rAND'): If $\mathcal{T}_1 \subseteq K(f_X(\mathcal{T})), \mathcal{T}_2 \in \mathcal{J}, \mathcal{T}_2 \subseteq f_X(\mathcal{T})$ then $\mathcal{T}_1 \sqcup \mathcal{T}_2 \subseteq f_X(\mathcal{T})$.

Example 2.2 shows that f_X does not necessarily satisfy (AND).

We list now a few properties of X-mappings. Firstly, as any X-mapping is an S-mapping, it possesses the following properties:

Property 2.11 Any S-mapping f satisfies (LLE), (RW), (rAND) and the property of *union of chains* (UC):

$$\text{If } \mathcal{T}_i \subseteq \mathcal{T}_{i+1} \subseteq f(\mathcal{T}), \mathcal{T}_i \in \mathcal{J} \text{ for any } i \in \mathbb{N}, \text{ then } \bigsqcup_{i \in \mathbb{N}} \mathcal{T}_i \subseteq f(\mathcal{T}).$$

<u>Proof:</u> (LLE), (RW) and (rAND): see above. (UC): see (prop$\sqcup - \subseteq$). □

We examine now some properties specific to X-mappings:

Property 2.12 1. If $X = \bigcup_{i \in I} X_i$, then $f_X(\mathcal{T}) = \bigcap_{i \in I} f_{X_i}(\mathcal{T})$.
Thus: If $X_1 \subseteq X_2$, then $f_{X_2}(\mathcal{T}) \subseteq f_{X_1}(\mathcal{T})$.

2. $\varphi \in f_X(\mathcal{T})$ iff $\neg\varphi \notin \bigcup_{x \in X - Th(\mathcal{T})}(\mathcal{T} \sqcup \neg x)$, i.e.
$$f_X(\mathcal{T}) = \neg\left(\bigcap_{x \in X - Th(\mathcal{T})} (\boldsymbol{\mathcal{L}} - (\mathcal{T} \sqcup \neg x))\right).$$

3. If $X \in \mathcal{S}$ or $(\boldsymbol{\mathcal{L}} - X)^\wedge = \boldsymbol{\mathcal{L}} - X$, then $Th \circ f_X = f_X$ (i.e. f_X is a pre-circumscription). Thus we get that if $\boldsymbol{\mathcal{L}} - X \in \mathcal{J}$, then $Th \circ f_X = f_X$, as noted in [SF96].

4. $\perp \in f_X(\mathcal{T})$ iff $f_X(\mathcal{T}) = \boldsymbol{\mathcal{L}}$ iff $X \subseteq Th(\mathcal{T})$.

5. $f_{X\cup\{x1,x2\}}(\mathcal{T}) \subseteq f_{X\cup\{x1\wedge x2\}}(\mathcal{T})$.
If $x1 \in \mathcal{T}$ or $x2 \in \mathcal{T}$, we get $f_{X\cup\{x1,x2\}}(\mathcal{T}) = f_{X\cup\{x1\wedge x2\}}(\mathcal{T})$.

Proof: Use property 2.3-2) in order to get points 2) and 4). 1) comes from 2). 3) comes directly from definition 2.1.

5) We suppose $\varphi \in f_{X\cup\{x_1,x_2\}}(\mathcal{T})$. From property 2.3-1) we get: for any $x \in X \cup \{x_1, x_2\}$, $x \in \mathcal{T}$ or $x \notin \mathcal{T} \sqcup \varphi$. If $x_1 \wedge x_2 \in \mathcal{T}$, from property 2.3-1) again, we get $\varphi \in f_{X\cup\{x_1\wedge x_2\}}$. Otherwise $x_1 \notin \mathcal{T}$ or $x_2 \notin \mathcal{T}$. Let us suppose $\mathbf{x}_1 \notin \mathcal{T}$, then $x_1 \notin \mathcal{T} \sqcup \varphi$, thus $x_1 \wedge x_2 \notin \mathcal{T} \sqcup \varphi$ and $\varphi \in f_{X\cup\{x_1\wedge x_2\}}(\mathcal{T})$.

Let us suppose now $x_1 \in \mathcal{T}$ or $x_2 \in \mathcal{T}$, say $x_1 \in \mathcal{T}$, and also $\varphi \in f_{X\cup\{x_1\wedge x_2\}}$. From property 2.3-1) we get $x_1 \wedge x_2 \in \mathcal{T}$ or $x_1 \wedge x_2 \notin \mathcal{T} \sqcup \varphi$. We consider $\{x_1, x_2\}$. As $x_1 \in \mathcal{T}$, we do not have problems here. If $x_2 \in \mathcal{T}$, then $\varphi \in f_{X\cup\{x_1,x_2\}}$ from property 2.3-1). Otherwise $x_1 \wedge x_2 \notin \mathcal{T}$ thus $x_1 \wedge x_2 \notin \mathcal{T} \sqcup \varphi$, thus $x_2 \notin \mathcal{T} \sqcup \varphi$ and again $\varphi \in f_{X\cup\{x_1,x_2\}}(\mathcal{T})$.□

For point 2), it is convenient to remind the following obvious results:

Property 2.13 • For any sets of formulas, we have: $\neg(X - Y) = \neg(X) - \neg(Y)$, $\neg\left(\bigcup_{i\in I} X_i\right) = \bigcup_{i\in I} \neg(X_i)$, $\neg\left(\bigcap_{i\in I} X_i\right) = \bigcap_{i\in I} \neg(X_i)$.

• For any $\mathcal{T} \in \mathcal{J}$, if the decomposition of $\mathcal{T}$ into complete theories is $\Gamma(\mathcal{T}) = \{\mathcal{T}'_i\}_{i\in I}$, we get: $\neg(\mathcal{T}) = \neg\left(\bigcap_{i\in I} \mathcal{T}'_i\right) = \boldsymbol{\mathcal{L}} - \bigcup_{i\in I} \mathcal{T}'_i$,

thus, as $\neg(\boldsymbol{\mathcal{L}}) = \boldsymbol{\mathcal{L}}$: $\neg(\boldsymbol{\mathcal{L}} - \mathcal{T}) = \neg\left(\boldsymbol{\mathcal{L}} - \bigcap_{i\in I} \mathcal{T}'_i\right) = \bigcup_{i\in I} \mathcal{T}'_i$.

Thus (see point 2) in property 2.12), when X is a singleton, $f_X(\mathcal{T})$ is either $\boldsymbol{\mathcal{L}}$ or a union of complete theories (as in example 2.2).

Theorem 2.14 $f_{X\cup\{x1,x2,x1\wedge x2\}}(\mathcal{T}) = f_{X\cup\{x1,x2\}}(\mathcal{T})$.

Proof: Use points 1) and 5) in property 2.12. This is the key result of this text, from which our other main results come easily. □

We give now a few properties of X-mappings, which have a meaning for any S-mapping, but which are not true for any S-mapping:

Theorem 2.15 In the following, X and $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ are elements of $\mathcal{F}$, while $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_i$ are elements of $\mathcal{J}$.

Any X-mapping f_X has the following properties of S-mappings:

1. *Strong cumulative transitivity*

   (sCT): If $\mathcal{T}_1 \subseteq f_X(\mathcal{T})$ and $\mathcal{T}_2 \subseteq f_X(\mathcal{T} \cup \mathcal{T}_1)$ then $\mathcal{T}_1 \sqcup \mathcal{T}_2 \subseteq f_X(\mathcal{T})$.

   Thus, we get a fortiori $\mathcal{T}_2 \subseteq f_X(\mathcal{T})$, which gives the more classic *cumulative transitivity*, also called *Cut* in the literature:

   (CT): If $\mathcal{T}_1 \subseteq f_X(\mathcal{T})$ then $f_X(\mathcal{T} \cup \mathcal{T}_1) \subseteq f_X(\mathcal{T})$.

   For S-mappings, (sCT) is (generally) strictly stronger than (CT), given in [SF96].

   Again, an consequence is the formula version of (CT), called here

   (CT1): If $\varphi \in f_X(\mathcal{T})$ and $\psi \in f_X(\mathcal{T} \sqcup \varphi)$ then $\psi \in f_X(\mathcal{T})$.

2. *Case reasoning (infinite version)* (CR∞): $\bigcap_{i \in I} f_X(\mathcal{T}_i) \subseteq f_X(\bigcap_{i \in I} \mathcal{T}_i)$.

   As a consequence, we get the finitary (i.e. binary) version

   (CR): $f_X(\mathcal{T}_1) \cap f_X(\mathcal{T}_2) \subseteq f_X(\mathcal{T}_1 \cap \mathcal{T}_2)$.

   Thus, we get the formula version, also given in [SF96],

   (CR1): If $\psi \in f_X(\mathcal{T} \sqcup \varphi_i)(i \in \{1,2\})$, then $\psi \in f_X(\mathcal{T} \sqcup (\varphi_1 \vee \varphi_2))$.

3. *Restricted cumulative monotony*

   (rCM): If $\mathcal{T} \subseteq \mathcal{T}' \subseteq \mathcal{T}_2 \subseteq f_X(\mathcal{T})$ then $\mathcal{T}_2 \subseteq f_X(\mathcal{T}')$.

   The formula version is (rCM1): $\psi \models \varphi, \psi \in f_X(\mathcal{T})$ imply $\psi \in f_X(\mathcal{T} \sqcup \varphi)$.

   This property is the classical cumulative monotony, but restricted to the subsets of $f_X(\mathcal{T})$ which are in $\mathcal{J}$. □

The proofs are immediate.

Any S-mapping satisfying (AND) and (rCM) satisfies classical *cumulative monotony*, i.e., for pre-circumscriptions, (rCM) is equivalent to
(CM): If $\mathcal{T} \subseteq \mathcal{T}' \subseteq f_X(\mathcal{T})$ then $f_X(\mathcal{T}) \subseteq f_X(\mathcal{T}')$.

Thus, as noted in [SF96], any X-mapping which is a pre-circumscription satisfies (CT) and (CM), that is it satisfies the now classical *cumulativity* (CUMU): If $\mathcal{T} \subseteq \mathcal{T}' \subseteq f_X(\mathcal{T})$ then $f_X(\mathcal{T}) = f_X(\mathcal{T}')$.

The expression introduced in [SF96] under the name "restricted cumulativity" is in fact only a variation around LLE (for one way of the "iff" given there, the "if" side of this "iff" being obscure for us).

As also noted in [SF96], X-mappings do not necessarily satisfy (CM). Example 2.2 is a good illustration of what happens:

f_X falsifies (CM): $\mathcal{T} = Th(\neg A \vee \neg B), \mathcal{T}' = Th(\neg B)$: $Th(\neg A \vee \neg B) \subseteq Th(\neg B) \subseteq f_X(\neg A \vee \neg B)$, while $f_X(\neg A \vee \neg B) \not\subseteq f_X(\neg B)$. Notice that it can be shown that an X-mapping satisfies (AND1) iff it satisfies (CM1).

Now, (rCM) is verified, because in (rCM) we are allowed to take only theories included in $f_X(\mathcal{T}) = f_X(\neg A \vee \neg B)$ and which also contain $\mathcal{T}' = Th(\neg B)$. All these theories are indeed included in $f_X(\neg B)$.

3 When X-mappings are pre-circumscriptions

Definitions 3.1 For any S-mapping f, we define the set of the fomulas *inaccessible for* f: $I_f = \boldsymbol{\mathcal{L}} - \bigcup_{\mathcal{T}\in\mathfrak{T}} (f(\mathcal{T}) - \mathcal{T}) = \bigcap_{\mathcal{T}\in\mathfrak{T}} (\boldsymbol{\mathcal{L}} - (f(\mathcal{T}) - \mathcal{T}))$.

For any set $X \in \mathfrak{F}$, we define $I_X = I_{f_X}$.

This set of inaccessible formulas play a fundamental role for X-mappings:

Theorem 3.2 1) For any S-mapping f, $I_f = (I_f)^\wedge$.

2) For any set X, $I_X = X$ iff $X = X^\wedge$.

3) For any $X \subseteq \mathfrak{F}$, $I_X = X^\wedge$.

Proof: 1) is obvious. 2) $X \subseteq I_X$ is obvious, the proof of the converse is longer and we give only indications here. Let us say that a set X is *maximum* iff it is the greatest (for $\subseteq$) set in $\{Y \in \mathfrak{F} / f_X = f_Y\}$ (such a set always exists from property 2.12-1). If X is maximum, then $X = X^\wedge$ (easy). It can be shown also that if X is maximum, then $I_X \subseteq X$, and conversely, that if $I_X = X$, then X is maximum. Finally, it can be shown that if $X = X^\wedge$, then X is maximum.

3) From 2) and theorem 2.14.

Notice that generally, I_X is different from $(I_X)^\vee$. Example: $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \{A, B\}$, $I_{\{A,B\}} = \{\top, A, B, A \wedge B\}$: $A \in I_{\{A,B\}}, B \in I_{\{A,B\}}, A \vee B \notin I_{\{A,B\}}$. $\square$

Definition 3.3 An S-mapping f *preserves the consistency* ((PC) for short) iff for any $\mathcal{T} \in \mathfrak{T}$, $f(\mathcal{T}) = \boldsymbol{\mathcal{L}} = Th(\perp)$ implies $\mathcal{T} = \boldsymbol{\mathcal{L}}$.

Property 3.4 An X-mapping satisfies (PC) iff $\perp \in X^\wedge$.

Proof: Use theorem 3.2 3) and property 2.3.

Theorem 3.5 In the finite propositional case, a pre-circumscription is an X-mapping iff it satisfies (CUMU) and (CR1).

Proof: We already know that any X-mapping which is a pre-circumscription satisfies (CUMU) and (CR∞), thus (CR1). f is a pre-circumscription satisfying (CUMU) and (CR1), and $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite (thus (CR1) is equivalent to

(CR∞)). Any ψ in $f(\mathcal{T})$ is easily proved to be in $f_{I_f}(\mathcal{T})$. Let us suppose that there exists some φ in $f_{I_f}(\mathcal{T}) - f(\mathcal{T})$, and that φ is "maximal" in this set (no $\psi \neq \varphi$ in this set satisfies $\psi \models \varphi$). For any formula θ such that $Th(\theta) = \mathcal{T} \sqcup \varphi$, we have $\theta \in f_{I_f}(\mathcal{T})$. Also, $\varphi \notin f(\mathcal{T})$ thus $\theta \notin f(\mathcal{T})$. From the "maximality" of φ, we get $\theta = \varphi$ thus $\mathcal{T} \subseteq Th(\varphi)$, i.e. $Th(\varphi) = \mathcal{T} \sqcup \varphi$.

As $\varphi \notin f(\mathcal{T})$, we get $\varphi \in \mathcal{T} \sqcup \varphi - \mathcal{T}$, thus $\varphi \in \boldsymbol{\mathcal{L}} - I_f$.

Let $\mathcal{T}''$ be some theory such that $\varphi \in f(\mathcal{T}'') - \mathcal{T}''$ and $\mathcal{T}' = \bigcap_{\varphi \in f(\mathcal{T}''') - \mathcal{T}'''} \mathcal{T}'''$. $\mathcal{T}' \subseteq \mathcal{T}''$, thus $\varphi \notin \mathcal{T}'$. From (CR1), $\varphi \in f(\mathcal{T}')$, thus $\mathcal{T}'$ is minimum (for $\subseteq$) in $\{\mathcal{T}''' \in \mathcal{J} / \varphi \in f(\mathcal{T}''') - \mathcal{T}'''\}$. $\varphi \in f(\varphi)$ thus $\varphi \in f(\mathcal{T}' \cap Th(\varphi))$. $\varphi \notin \mathcal{T}' \cap Th(\varphi)$, thus $\mathcal{T}' \subseteq \mathcal{T}' \cap Th(\varphi)$, i.e. $\mathcal{T}' \subseteq Th(\varphi)$.

Let ψ be such that $Th(\psi) = \mathcal{T}'$, we get $\psi \in \mathcal{T} \sqcup \varphi$. If $\psi \in \mathcal{T}$, $\mathcal{T}' \subseteq \mathcal{T} \subseteq \mathcal{T} \sqcup \varphi = Th(\varphi) \subseteq f(\mathcal{T}')$: from (CUMU), we get $f(\mathcal{T}) = f(\mathcal{T}')$ and $\varphi \in f(\mathcal{T})$, a contradiction. Thus $\psi \in \mathcal{T} \sqcup \varphi - \mathcal{T}$, and $\psi \in \boldsymbol{\mathcal{L}} - I_f$.

Let $\mathcal{T}''' \in \mathcal{J}$ be such that $\psi \in f(\mathcal{T}''') - \mathcal{T}'''$, we get $\mathcal{T}' \cap \mathcal{T}''' \subseteq \mathcal{T}' \subseteq f(\mathcal{T}' \cap \mathcal{T}''')$. Thus $f(\mathcal{T}') = f(\mathcal{T}' \cap \mathcal{T}''')$. $\varphi \notin \mathcal{T}' \cap \mathcal{T}'''$ thus $\mathcal{T}' \cap \mathcal{T}''' = \mathcal{T}'$ (as $\mathcal{T}'$ is minimum), thus $\psi \in \mathcal{T}'''$, a contradiction. Thus $f_{I_f}(\mathcal{T}) = f(\mathcal{T})$. $\square$

This result does not extend to the infinite case: there exist pre-circumscriptions satisfying (CUMU) and (CR∞) which are not X-mappings.

4 Preferential entailments

Definition 4.1 • A *preference relation* (pr for short) $\prec$ is a binary relation over $\mathcal{C}$. For any $\mathcal{T} \in \mathcal{J}$, we define the set $\Gamma_\prec(\mathcal{T}) = \{\mathcal{T}' \in \Gamma(\mathcal{T}) /$ for no $\mathcal{T}'' \in \Gamma(\mathcal{T})$, we have $\mathcal{T}'' \prec \mathcal{T}\}$. A *preferential entailment* (pe for short) $f_\prec$ is the pre-circumscription defined by $f_\prec(\mathcal{T}) = \bigcap_{\mathcal{T}' \in \Gamma_\prec(\mathcal{T})} \mathcal{T}'$.

• m($\mathcal{C}$) is some set of "copies" of elements of $\mathcal{C}$ (if $\mathcal{C}$ is finite, we allow only finitely many copies $\mathcal{T}'^i$ for each element $\mathcal{T}'$ of $\mathcal{C}$). A *multi preference relation* (mpr) $\prec$ is a binary relation over m($\mathcal{C}$). For any $\mathcal{T} \in \mathcal{J}$, we define the set $\Gamma_\prec(\mathcal{T}) = \{\mathcal{T}' \in \Gamma(\mathcal{T}) /$ there exists some copy $\mathcal{T}'^i$ of $\mathcal{T}'$ in m($\mathcal{C}$) such that for no copy $\mathcal{T}''^j$ of some $\mathcal{T}'' \in \Gamma(\mathcal{T})$, we have $\mathcal{T}''^j \prec \mathcal{T}'^i\}$.

A *multi preferential entailment* (mpe) is a pre-circumscription $f_\prec$ defined by: $f_\prec(\mathcal{T}) = \bigcap_{\mathcal{T}' \in \Gamma_\prec(\mathcal{T})} \mathcal{T}'$. $\square$

Thus we get, for a pe or a mpe $f_\prec$: $\Gamma(f_\prec(\mathcal{T})) = \Phi(\Gamma_\prec(\mathcal{T}))$. Φ denotes the classical topological closure over $\mathcal{C}$: for any $\boldsymbol{\mathcal{D}} \subseteq \mathcal{C}$, $\Phi(\boldsymbol{\mathcal{D}}) = \Gamma(\bigcap_{\mathcal{T}' \in \boldsymbol{\mathcal{D}}} \mathcal{T}')$.

When $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite: $\Gamma(f_\prec(\mathcal{T})) = \Gamma_\prec(\mathcal{T})$.

When a result about pe or mpe is given as "known", it comes from one of the following texts [Sho88, KLM90, Sat90, LM92, FL93, Mak94, MR94a, MR94b, Sch97], which are some of the classical references on the subject. The results useful for circumscriptions are completed and precised in [MR97]. [KLM90] and its followers have introduced mpe's in order to simulate some aspects of pe's of predicate logic in a propositional framework.

Property 4.2 The following results are known:
1) Any mpe satisfies (CT) and (CR). Any pe is a mpe.
A pe may falsify (CR∞) or (CM1).
2) Any pe satisfies (DCC): $f(\mathcal{T}) \subseteq \bigcap_{\mathcal{T}' \in \Gamma(\mathcal{T})} \bigsqcup_{\mathcal{T}'' \in \Gamma(\mathcal{T})} f(\mathcal{T}' \cap \mathcal{T}'')$.
3) Let (DC1) be: $f(\mathcal{T} \sqcup (\varphi \vee \psi)) \subseteq f(\mathcal{T} \sqcup \varphi) \sqcup f(\mathcal{T} \sqcup \psi)$.
If $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is infinite, a pe may falsify (DC1). If $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite, a pre-circumscription is an mpe iff it satisfies (CR1) and a pre-circumscription is a pe iff it satisfies (CR1) and (DCC), iff it satisfies (CR1) and (DC1).

Definition 4.3 A pr, or a mpr is *quasi safely founded* (qsf for short) iff, for any $\mathcal{T} \in \mathcal{J}$, for any $\mathcal{T}' \in \Gamma(\mathcal{T})$, $\mathcal{T}' \prec \mathcal{T}'$ or $\mathcal{T}' \in \Gamma_{\prec}(\mathcal{T})$ or there exists $\mathcal{T}'' \in \Gamma_{\prec}(\mathcal{T})$ such that $\mathcal{T}'' \prec \mathcal{T}'$.

This is related to a notion called well-founded, stoppered or smooth in the literature.

Property 4.4 It is known that if a mpr satisfies qsf then the mpe $f_{\prec}$ satisfies (CUMU). In the finite case, we get the converse: if a mpe f satisfies (CUMU), then there exists some mpr $\prec$ satisfying qsf such that $f = f_{\prec}$.

Definition 4.5 The reduced form $\prec_r$ of a pr $\prec$ is defined by: $\mathcal{T}' \prec_r \mathcal{T}''$ iff $\mathcal{T}' \prec \mathcal{T}''$ and ($\mathcal{T}'' \not\prec \mathcal{T}''$ or $\mathcal{T}' = \mathcal{T}''$).

It is obvious that we get $f_{\prec} = f_{\prec_r}$. This is also true for an mpr $\prec$ but it is not really interesting. Indeed, (at least in the finite case) for two pr's $\prec$ and $\prec'$ such that $f_{\prec} = f_{\prec'}$, we get $\prec_r = \prec'_r$, but this result is false when $\prec$ and $\prec'$ are arbitarry mpr's. A reduced form, with such a uniqueness result, may be defined for mpr's also, but this definition is more technical and we have no room to put it here. Notice however that we will describe a method which allows to find this reduced form (in the finite cumulative cases).

Property 4.6 1) A pr $\prec$ is qsf iff $\prec_r$ is qsf and transitive and with *unitary loops isolated*. This last expression means that if $\mathcal{T}' \prec_r \mathcal{T}'$, then for any $\mathcal{T}'' \in \mathcal{C}$, if $\mathcal{T}'' \neq \mathcal{T}'$, then $\mathcal{T}'' \not\prec_r \mathcal{T}'$ and $\mathcal{T}' \not\prec_r \mathcal{T}''$.

2) In the finite case, a pr $\prec$ is qsf iff $\prec_r$ is transitive and with unitary loops isolated (then $\prec_r$ is necessarily qsf).

3) Similar results exist for mpr's, which we do not explicit here.

The proof of this result, which is only a marginal extension of already known results, is easy and omitted here.

In the finite case, a pe $f_\prec$ is (PC) iff $\prec$ has no loop $\mathcal{T}'_1 \prec \mathcal{T}'_2 \prec \cdots \prec \mathcal{T}'_n \prec \mathcal{T}'_1$, iff $\prec_r$ has no loop.

Thus we get:

Property 4.7 (known) In the finite case, a pe $f_\prec$ satisfies (CUMU) and (PC) iff $\prec_r$ is transitive and irreflexive ($\mathcal{T}' \not\prec_r \mathcal{T}'$) (then $\prec_r$ must be qsf).

We give now easy results which are not classical in the literature.

Definition 4.8 A subset $\mathcal{C}_1$ of $\mathcal{C}$ is *finishing for a pr* $\prec$ iff:
1) for any $\mathcal{T}' \in \mathcal{C}$, if $\mathcal{T}' \prec_r \mathcal{T}'$, then $\mathcal{T}' \in \mathcal{C}_1$ and
2) for any $\mathcal{T}'' \in \mathcal{C}$, any $\mathcal{T}' \in \mathcal{C}_1$, if $\mathcal{T}' \prec_r \mathcal{T}''$, then $\mathcal{T}'' \in \mathcal{C}_1$.

Property 4.9 If a pr $\prec$ is qsf, then for any φ, $\Gamma(\varphi)$ is finishing for $\prec$ iff $\varphi \in I_{f_\prec}$ (φ inaccessible for $f_\prec$).

Proof: $\Gamma(\varphi)$ is finishing for $\prec$, $\mathcal{T} \in \mathcal{J}$. We suppose that there exists $\mathcal{T}' \in \Gamma(\mathcal{T}) - \Gamma(\varphi)$. Thus, $\mathcal{T}' \not\prec_r \mathcal{T}'$ (definition 4.8). From qsf, there exists $\mathcal{T}'' \in \Gamma_\prec(\mathcal{T})$ such that $\mathcal{T}' = \mathcal{T}''$ or $\mathcal{T}'' \prec_r \mathcal{T}'$. As $\Gamma(\varphi)$ is finishing, we get $\mathcal{T}'' \notin \Gamma(\varphi)$. We have established: if $\Gamma(\mathcal{T}) \not\subseteq \Gamma(\varphi)$, then $\Gamma_\prec(\mathcal{T}) \not\subseteq \Gamma(\varphi)$, i.e. $\Gamma(f_\prec(\mathcal{T})) \not\subseteq \Gamma(\varphi)$ (indeed, $\Gamma(\varphi) = \Phi(\Gamma(\varphi))$). This means: if $\varphi \notin \mathcal{T}$, then $\varphi \notin f_\prec(\mathcal{T})$. This establishes $\varphi \in I_{f_\prec}$.

If now $\Gamma(\varphi)$ is not finishing for $\prec$, there exists $\mathcal{T}' \in \mathcal{C} - \Gamma(\varphi)$ such that, either $\mathcal{T}' \prec_r \mathcal{T}'$, or there exists $\mathcal{T}'' \in \Gamma(\varphi)$ such that $\mathcal{T}'' \prec_r \mathcal{T}'$. We define $\mathcal{T} = \mathcal{T}' \cap Th(\varphi)$, ie $\Gamma(\mathcal{T}) = \Gamma(\varphi) \cup \{\mathcal{T}'\}$. As $\mathcal{T}' \notin \Gamma(\varphi)$, we get $\mathcal{T} \neq Th(\varphi)$: $\mathcal{T} \subset Th(\varphi)$. As $\mathcal{T}' \prec_r \mathcal{T}'$ or $\mathcal{T}'' \prec_r \mathcal{T}'$, we get: $\Gamma_\prec(\mathcal{T}) = \Gamma_\prec(\varphi) \subseteq \Gamma(\varphi)$. Thus, $\Gamma(f_\prec(\mathcal{T})) \subseteq \Gamma(\varphi)$, i.e. $\varphi \in f_\prec(\mathcal{T})$, and also $\varphi \notin \mathcal{T}$: $\varphi \notin I_{f_\prec}$. $\square$

Notice that if $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \{A_i\}_{i\in I}$ and $f_\prec$ is the parallel circumscription of all the A_i's, the set $I_{f_\prec}$ is the set of all the formulas *positive in* $\{A_i\}_{i\in I}$. In fact, we have defined, with $I_{f_\prec}$, what could be called the set of the formulas *positive for* $f_\prec$, for any cumulative pe $f_\prec$.

We get also:

Property 4.10 The set of the formulas inaccessible for a cumulative pe $f_\prec$ is closed for $\vee$. As we already know that it is closed for $\wedge$ (theorem 3.2-1), we get $(I_{f_\prec})^{\wedge\vee} = I_{f_\prec}$.

Proof: $\Gamma(\varphi_1 \wedge \varphi_2) = \Gamma(\varphi_1) \cap \Gamma(\varphi_2)$ and $\Gamma(\varphi_1 \vee \varphi_2) = \Gamma(\varphi_1) \cup \Gamma(\varphi_2)$. It is obvious from definition 4.8 that finishing sets for a pr $\prec$ which is qsf are stable for unions and disjunctions. □

The interest of the study of inaccessible formulas comes from theorem 3.2. The interest of our study of reduced relations will come now. Notice that this method of studying cumulative preferential entailments through their inaccessible formulas is somehow paradoxical: generally, we are more interested in formulas that a pe may give. Also, this is an illustration of the power of the condition of cumulativity: for any pre-circumscription, in the finite case, when f satisfies (CUMU) and (CR1), then f is fully described only from its inaccessible formulas. Notice that in the infinite case, an example (omitted here) shows that we need more than (CUMU) + (CR∞) + (DCC) + full *reverse monotony* (RM) (called infinite conditionalization in e.g. [Sch97]), to be sure to get an X-mapping.

(RM): $f(\mathcal{T}, \mathcal{T}'') \subseteq f(\mathcal{T}) \sqcup \mathcal{T}''$ for any $\mathcal{T}, \mathcal{T}''$ in $\mathcal{F}$.

Another example shows that X-mappings may falsify full (RM).

5 The finite case

We restrict ourselves to the finite case ($\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite), in order to get more precise results in the relations between a cumulative pe or mpe and its corresponding X-mapping. Generally, the definitions and results of this section do not extend to the infinite case.

Definition 5.1 A formula φ is *complete* iff $Th(\varphi) \in \mathcal{C}$.

A formula φ is a *minimal clause* iff $\neg\varphi$ is complete.

To any element $\mathcal{T}' \in \mathcal{C}$ we associate a (complete) formula $\phi(\mathcal{T}')$ such that $Th(\phi(\mathcal{T}')) = \mathcal{T}'$.

We need the following definition, together with an easy result, probably well-known in classical finite logic:

Definition 5.2 For any subset X of $\mathcal{F}$, we define the set $X_\wedge = \{x \in X / x \notin (X - \{x\})^\wedge\}$.

Property 5.3 For any subset X of $\mathcal{F}$ which is closed for $\wedge$, $X_\wedge$ is the smallest (for $\subseteq$) set Y such that $X = Y^\wedge$. We call $X_\wedge$ the $\wedge$-*basis* of X.

Moreover we have, for any subsets X and Y of $\mathcal{F}$, $X^\wedge = Y^\wedge$ iff $X_\wedge = Y_\wedge$.

From theorem 3.2 we get:

Property 5.4 $f_X = f_Y$ iff $X_\wedge = Y_\wedge$ iff $X_\wedge \subseteq Y \subseteq X^\wedge$.

This gives a way to find the smallest possible set X which may be used to define any given X-mapping: we need only to calculate $X_\wedge$.

Definitions 5.5 Let us suppose that $\prec$ is some pr satisfying qsf. For any $\mathcal{T}' \in \mathcal{C}$, we define the set of the predecessors of $\mathcal{T}'$:

$C_\prec(\mathcal{T}') = \{\mathcal{T}'' \in \mathcal{C} / \mathcal{T}'' \prec_r \mathcal{T}' \text{ or } \mathcal{T}'' = \mathcal{T}'\}$.

To any $\mathcal{T}' \in \mathcal{C}$ such that $\mathcal{T}' \not\prec_r \mathcal{T}'$, we associate the formula $x(\mathcal{T}')$ such that $\Gamma(x(\mathcal{T}')) = \mathcal{C} - C_\prec(\mathcal{T}')$, i.e. $\Gamma(\neg x(\mathcal{T}')) = C_\prec(\mathcal{T}')$, i.e

$$x(\mathcal{T}') = \neg(\phi(\mathcal{T}') \vee \bigvee_{\mathcal{T}'' \prec_r \mathcal{T}'} \phi(\mathcal{T}'')).$$

We get a set $X_\prec = \{x(\mathcal{T}')/\mathcal{T}' \in \mathcal{C}, \mathcal{T}' \not\prec_r \mathcal{T}'\}$ of formulas which is equipotent to the set of the elements $\mathcal{T}'$ of $\mathcal{C}$ such that $\mathcal{T}' \not\prec \mathcal{T}'$ (indeed, it is easy to show that the $x(\mathcal{T}')$'s are all different, and $\mathcal{T}' \prec \mathcal{T}'$ iff $\mathcal{T}' \prec_r \mathcal{T}'$).

Property 5.6 For any pr $\prec$ which is qsf, we have $f_\prec = f_{X_\prec}$.

<u>Proof:</u> It suffices to show that $X_\prec$ is a $\wedge$-basis of $I_{f_\prec}$. That is, it suffices to show that the sets $\mathcal{C} - C_\prec(\mathcal{T}')$ constitute a "$\cap$-basis" (defined similarly to the $\wedge$-basis of definition 5.2) of the set of all the subsets of $\mathcal{C}$ which are finishing for $\prec$. It is obvious from the definitions that any $\mathcal{C} - C_\prec(\mathcal{T}')$ is finishing. It is also easy to prove that these sets are all the finishing subsets of $\mathcal{C}$ which are maximal (for $\subseteq$). It is longer, but not really difficult, to prove that any finishing subset of $\mathcal{C}$ is equal to some $\bigcap_{i \in I}(\mathcal{C} - C_\prec(\mathcal{T}'_i))$ where all the $\mathcal{T}'_i$'s are in $\mathcal{C}$ (if $I = \emptyset$, the intersection is $\mathcal{C}$). $\square$

Notice that not only we have found a set $X_\wedge$, but this set is structured, by $\models$, exactly as $\mathcal{C}$ is structured by $\succ_r$, the converse relation of $\prec_r$. This shows the importance of transitivity for $\prec_r$, as $\models$ is transitive.

Indeed, from above formulation we can see that we have:

Property 5.7 For any $\mathcal{T}'_1$, $\mathcal{T}'_2$ in $\mathcal{C}$ such that $\mathcal{T}'_1 \neq \mathcal{T}'_2$, $\mathcal{T}'_1 \not\prec_r \mathcal{T}'_1$, and $\mathcal{T}'_2 \not\prec_r \mathcal{T}'_2$, we have:

$$\mathcal{T}'_1 \prec_r \mathcal{T}'_2 \qquad \text{iff} \qquad x(\mathcal{T}'_2) \models x(\mathcal{T}'_1).$$

A nice thing with this result is that not only we obtain some set X, but we obtain, with $X_\prec$, the smallest possible set X such that $f_\prec = f_X$.

Example 5.8 $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \{A, B\}$, $\mathcal{T}'_1 = Th(A \wedge B)$, $\mathcal{T}'_2 = Th(A \wedge \neg B)$, $\mathcal{T}'_3 = Th(\neg A \wedge B)$, $\mathcal{T}'_4 = Th(\neg A \wedge \neg B)$. The graph of $\prec$ is $\{(\mathcal{T}'_1, \mathcal{T}'_1), (\mathcal{T}'_3, \mathcal{T}'_1), (\mathcal{T}'_3, \mathcal{T}'_2), (\mathcal{T}'_4, \mathcal{T}'_2)\}$. The graph of $\prec_r$ is the same less $(\mathcal{T}'_3, \mathcal{T}'_1)$. $\prec_r$ is transitive and its only loop is isolated.

We get $x(\mathcal{T}'_3) = \neg(\neg A \wedge B) = A \vee \neg B$, $x(\mathcal{T}'_4) = \neg(\neg A \wedge \neg B) = A \vee B$, and $x(\mathcal{T}'_2) = \neg((A \wedge \neg B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B) = A \wedge B$.

Thus $X_{\prec} = \{A \vee \neg B, A \vee B, A \wedge B\}$ and $I_{f_{\prec}} = (X_{\prec})^{\wedge} = \{\top, A \vee \neg B, A \vee B, A, A \wedge B\}$. This set is $\vee$-closed.

It is easy to check that $f_{X_{\prec}} = f_{\prec}$. And the relation $\prec_r$ is mirrored in $X_{\prec}$ by $\mathcal{T}'_i \prec_r \mathcal{T}'_j$ iff either $\mathcal{T}'_i \neq \mathcal{T}'_j$ and $x(\mathcal{T}'_j) \models x(\mathcal{T}'_i)$ in $X_{\prec}$ or there is no $x(\mathcal{T}'_i)$ in $X_{\prec}$ and $\mathcal{T}'_i = \mathcal{T}'_j$ (case where $i = j = 1$). □

And we see now how we can do the reverse operation: Starting from some set X we want to know if f_X is a pe, and if this is the case, we want to define some $\prec$ such that $f_X = f_{\prec}$.

It is natural then to introduce the following definitions:

Definition 5.9 To each $x \in X_{\wedge}$, where X is some subset of $\mathcal{F}$, we associate the formula

$$\varphi(x) = \neg x \wedge (\bigwedge_{y \in X_{\wedge},\, y \neq x,\, x \models y} y).$$

From the way we have calculated $X_{\wedge}$ associated to some given pe, it can be proved the following:

Property 5.10 1) An X-mapping f_X is a pe iff for any $x \in X_{\wedge}$, $y \in X_{\wedge}$:

C1) $\varphi(x)$ is a complete formula, and

C2) if $\varphi(x) = \varphi(y)$, then $x = y$.

2) If these two conditions are fulfilled, we define the following relation $\prec$ over $\mathcal{C}$: $\mathcal{T}'_1 \prec \mathcal{T}'_2$ iff

$\mathcal{T}'_1 = \mathcal{T}'_2$ and $\mathcal{T}'_1$ is different from all the $\varphi(x)$, for any $x \in X_{\wedge}$, or

there exist $x \neq y$ in $X_{\wedge}$ such that $\mathcal{T}'_1 = Th(\varphi(y)), \mathcal{T}'_2 = Th(\varphi(x)), x \models y$.

3) For any set X satisfying C1), we have C2) iff $X^{\wedge}$ is closed for $\vee$.

Indeed, if, from the pe $f_{\prec}$ defined in 2), we do the construction made in definitions 5.5, we recover the unique set $X_{\wedge} = X_{\prec}$ such that $f_{\prec} = f_{X_{\wedge}}$. Also, if some $\varphi(x)$ is not a complete formula, or if two different elements x, y of $X_{\wedge}$ give the same $\varphi(x)$, we cannot do what we have done above, which proves that the X-mapping cannot correspond to a pe. The complete proof is lengthy, but we hope to have given here the key points. For 3), we already know one way from property 4.10, the other way is slightly more tricky. □

Example 5.11 $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \{A, B\}$. We start from $X = \{A \vee \neg B, A \vee B, A, A \wedge B\}$. We get $X_{\wedge} = \{A \vee \neg B, A \vee B, A \wedge B\}$.

Thus, we get $\varphi(A \vee \neg B) = \neg(A \vee \neg B) = \neg A \wedge B$, $\varphi(A \vee B) = \neg(A \vee B) = \neg A \wedge \neg B$, $\varphi(A \wedge B) = \neg(A \wedge B) \wedge ((A \vee \neg B) \wedge (A \vee B)) = A \wedge \neg B$. C1) and C2) are satisfied, thus we are in a case when f_X is a pe. And we may reconstruct $\prec_r$ in the following way: $Th(\neg A \wedge B) \prec_r Th(A \wedge \neg B)$ because $\varphi(A \vee \neg B) = \neg A \wedge B$, $\varphi(A \wedge B) = A \wedge \neg B$ and $A \wedge B \models A \vee \neg B$. Also $Th(A \wedge \neg B) \prec_r Th(A \wedge \neg B)$. Finally, as $Th(A \wedge B)$ is not $\varphi(x)$ for some $x \in X_\wedge$, we have also $Th(A \wedge B) \prec_r Th(A \wedge B)$. We find back the pe of example 5.8. □

Example 5.12 $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \{A, B\}$. $X = \{\bot, A \wedge B, A, B\}$. $X_\wedge = \{\bot, A, B\}$. $\varphi(A) = \neg A, \varphi(B) = \neg B, \varphi(\bot) = \neg\bot \wedge (A \wedge B) = A \wedge B$. C1 is falsified and indeed, f_X is not a pe, it is even not a pre-circumscription (see property 5.14 below). It can be calculated that we get indeed $f_X(\neg A \vee \neg B) = f_X(\top) = Th(\neg A \wedge \neg B) \cup Th(A \not\Leftrightarrow B)$ which is not a theory. □

Example 5.13 $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \{A, B\}$. $X = \{A \vee B, A \vee \neg B, \neg A \vee \neg B, \neg A \wedge B, \neg A \wedge \neg B)\}$. $X = X_\wedge$. $X^\wedge = \{\top, A \vee B, A \vee \neg B, \neg A \vee \neg B, A, \neg B, A \not\Leftrightarrow B, A \wedge \neg B, \neg A \wedge B, \neg A \wedge \neg B, \bot)$.

Calculating all the $\varphi(x)$ when $x \in X_\wedge$, we get:

$\varphi(A \vee B) = \neg A \wedge \neg B$, $\varphi(A \vee \neg B) = \neg A \wedge B$, $\varphi(\neg A \vee \neg B) = A \wedge B$, $\varphi(\neg A \wedge B) = A \wedge \neg B$, $\varphi(\neg A \wedge \neg B) = A \wedge \neg B$. Here C1 is satisfied, and indeed, f is a pre-circumscription (see below). However, C2) is falsified: $\varphi(\neg A \wedge B) = \varphi(\neg A \wedge \neg B)$, and it can be shown that f_X is indeed not a pe: it falsifies (DC1): It can be computed (not so easily, see below for an easier way) that $f_X(\neg B) = Th(\neg B), f_X(A \not\Leftrightarrow B) = Th(A \not\Leftrightarrow B)$, while $Th(\neg B) \cap Th(A \not\Leftrightarrow B) = Th(\neg A \vee \neg B)$ and $f_X(\neg A \vee \neg B) = Th(\neg A) \not\subseteq Th((\neg B) \wedge A \not\Leftrightarrow B) = Th(A \wedge \neg B)$. □

Thanks to the notion of mpe, we may extend this method to pre-circumscriptions, and not only pe's. Indeed, in the finite case a pre-circumscription satisfies (CR1) iff it is an mpe. The difficulty is to define the (pertinent) reduced form of a mpr $\prec$. But this can be done, and, with some complications in all the definitions (finishing sets, $C_\prec$,...) the same method works. In particular, the formula $\varphi(x)$ has exactly the same definition. We have no room to detail it here, but it works, and the transformations formulas are the same, once the definitions have been made precise.

The result, as the reader can guess at this step, is the following:

Property 5.14 • An X-mapping f_X is a pre-circumscription iff C1) of property 5.10 is satisfied.

• If (C1) is satisfied, then we define the following relation $\prec$ over some set of copies of $\mathcal{C}$:

$\mathcal{T'}_i^j \prec \mathcal{T'}_k^l$ iff these two copies are identical and are a copy of some $\mathcal{T}'_i$ which is different from all the $\varphi(x)$, for any $x \in X_\wedge$, or
there exist $x \neq y$ in $X_\wedge$ such that $\mathcal{T'}_i^j = Th(\varphi(y)), \mathcal{T'}_k^l = Th(\varphi(x)), x \models y$.

Example 5.15 We come back to example 5.13. From our computation of the $\varphi(x)$, if we define the $\mathcal{T}'_i$'s as in example 5.8, we get the following set m($\mathcal{C}$)= $\{\mathcal{T}'_1, \mathcal{T}'_3, T'_4, (\mathcal{T}'_2)^1, (\mathcal{T}'_2)^2\}$, with $Th(\varphi(A \vee B)) = \mathcal{T}'_4$, $Th(\varphi(A \vee \neg B)) = \mathcal{T}'_3$, $Th(\varphi(\neg A \vee \neg B)) = \mathcal{T}'_1$, $Th(\varphi(\neg A \wedge B)) = \mathcal{T}'_2$, first copy which will be called $(\mathcal{T}'_2)^1$, $Th(\varphi(\neg A \wedge \neg B)) = \mathcal{T}'_2$, second copy which will be called $(\mathcal{T}'_2)^2$. Thus we get the following mpr $\prec$:

$\mathcal{T}'_4 \prec (\mathcal{T}'_2)^1$, $\mathcal{T}'_1 \prec (\mathcal{T}'_2)^1$, $\mathcal{T}'_3 \prec (\mathcal{T}'_2)^2$, $\mathcal{T}'_1 \prec (\mathcal{T}'_2)^2$. Here is one example showing how to find this relation: $\mathcal{T}'_4 = Th(\varphi(A \vee B)), (\mathcal{T}'_2)^1 = Th(\varphi(\neg A \wedge B))$, and $\neg A \wedge B \models A \vee B$, thus $\mathcal{T}'_4 \prec (\mathcal{T}'_2)^1$.

It may be checked that $f_X = f_\prec$. Manually, it is easier to compute $f_\prec$ than f_X, and the reader may check the value of $f_\prec(\neg B)$ in this way: $\Gamma(\neg B) = \{\mathcal{T}'_4, \mathcal{T}'_2\}$, where $\mathcal{T}'_2$ comes with two copies. $\Gamma_\prec(\neg B) = \{\mathcal{T}'_4, \mathcal{T}'_2\}$ because the copy $(\mathcal{T}'_2)^2$ is not eliminated by $\mathcal{T}'_4$. Similarly $\Gamma_\prec(A \not\Leftrightarrow B) = \Gamma(A \not\Leftrightarrow B) = \{\mathcal{T}'_3, \mathcal{T}'_2\}$ because the copy $(\mathcal{T}'_2)^1$ is not eliminated by $\mathcal{T}'_3$.

However, $\Gamma(\neg A \vee \neg B) = \{\mathcal{T}'_2, \mathcal{T}'_3, \mathcal{T}'_4\}$ and $\Gamma_\prec(\neg A \vee \neg B) = \{\mathcal{T}'_3, \mathcal{T}'_4\}$: now the two copies of $\mathcal{T}'_2$ are eliminated (one by $\mathcal{T}'_3$, the other one by $\mathcal{T}'_4$).

This explains why (DC1) is violated, and why this is directly linked to the falsification of C2).

As a last remark about this example, notice that $X^\wedge$ is not $\vee$-closed.

6 A last example: traditional circumscriptions

We apply now our results to traditional circumscriptions (traditional except that we are in the finite propositional case). Predicate circumscription (see e.g [McC80, PM86, Lif94]) has already been simplified into "propositional circumscription" by many authors (e.g [EG93]).

We assume familiarity with this now well known notion.

Definition 6.1 A circumscription is defined once we know the set $\mathfrak{A}$ of the *circumscribed* propositional symbols, the set $\mathfrak{A}'$ of the *varying* ones, the remaining propositional symbols of $\mathcal{V}(\boldsymbol{\mathcal{L}})$, in $\mathfrak{A}''$, being *fixed*. $\mathcal{V}(\boldsymbol{\mathcal{L}}) = \mathfrak{A} \cup \mathfrak{A}' \cup \mathfrak{A}''$ (disjoint union).

The circumscription $CIRC(\mathfrak{A}, \mathfrak{A}', \mathfrak{A}'')$ is defined as the pe $f_\prec$ where $\mathcal{T}'_i \prec \mathcal{T}'_j$ iff $\{A \in \mathfrak{A} / \mathcal{T}'_i \models A\} \subset \{A \in \mathfrak{A} / \mathcal{T}'_j \models A\}$ and $\{A'' \in \mathfrak{A}'' / \mathcal{T}'_i \models A''\} = \{A'' \in \mathfrak{A}'' / \mathcal{T}'_j \models A''\}$.

$\prec$ is transitive, irreflexive and qsf (even in the infinite case). Thus, this kind of pe preserves consistency and is cumulative. Thus, when $\mathcal{V}(\boldsymbol{\mathcal{L}})$ is finite, we may always find some f_X associated to it.

Notice that $\prec$ is trivially in reduced form here, as there are no loops.

The problem is to find the finishing sets, i.e to find the formulas $x(\mathcal{T}')$.

We give here the results, as the technical details are rather lengthy. We must refer to the definition of $x(\mathcal{T}')$, but it may help also to remind definitions 4.8 and 5.5.

We suppose, for simplifying things, that $\mathcal{A} = \{A, B, C\}$, $\mathcal{A}' = \{D, E\}$ and $\mathcal{A}'' = \{F, G\}$. Generally, let us say that the respective cardinals of $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ are n, n', n''.

We examine the $\mathcal{T}_i'$'s "level by level", begining by the lowest level which corresponds to theories $\mathcal{T}' \in \mathcal{C}$ like $Th(\neg A \wedge \neg B \wedge \neg C \wedge \psi' \wedge \psi'')$, where ψ' and ψ'' are any complete formulas in the vocabulary of $\mathcal{A}'$ and $\mathcal{A}''$ respectively. This gives $x(\mathcal{T}') = \neg\phi(\mathcal{T}') = A \vee B \vee C \vee \neg\psi' \vee \neg\psi''$.

We get sixteen formulas ($16 = 2^{n'} \times 2^{n''}$), corresponding to all the clauses $\neg\psi'$ and $\neg\psi''$ minimal respectively for $\mathcal{A}'$ and $\mathcal{A}''$: $\neg\psi' = \epsilon D \vee \epsilon E$, $\neg\psi'' = \epsilon F \vee \epsilon G$, where ϵ is either nothing or $\neg$.

The level immediatly above consists of theories $\mathcal{T}' \in \mathcal{C}$ such as $Th(\neg A \wedge \neg B \wedge C \wedge \psi' \wedge \psi'')$. Now $x(\mathcal{T}') = \neg(\phi(\mathcal{T}') \vee (\neg A \wedge \neg B \wedge \neg C \wedge \psi'')) = A \vee B \vee (C \wedge \neg\psi') \vee \neg\psi''$. ψ' and ψ'' are as above. However now we get three $(3 = n)$ choices for the proposition in $\mathcal{A}$ which is not negated in $\mathcal{T}'$ (and which will be conjuncted with $\neg\psi'$) thus we get $n \times 2^{n'} \times 2^{n''}$ such formulas.

The next level consists of theories $\mathcal{T}' \in \mathcal{C}$ such as $Th(\neg A \wedge B \wedge C \wedge \psi' \wedge \psi'')$. Now $x(\mathcal{T}') = \neg(\phi(\mathcal{T}') \vee (\neg A \wedge (\neg B \vee \neg C) \wedge \psi'')) = A \vee (B \wedge C \wedge \neg\psi') \vee \neg\psi''$. ψ' and ψ'' are as above. We get three $(3 = (n, 2))$ choices for the two propositions in $\mathcal{A}$ which are not negated in $\mathcal{T}'$, thus we get $(n, 2) \times 2^{n'} \times 2^{n''}$ such formulas, with $(n, 2) = (n \times (n-1))/2$.

The next level, which is here the last level, consits of theories $\mathcal{T}'$ of the kind $Th(A \wedge B \wedge C \wedge \psi' \wedge \psi'')$. We get $x(\mathcal{T}') = \neg(\phi(\mathcal{T}') \vee ((\neg A \vee \neg B \vee \neg C) \wedge \psi'')) = (A \wedge B \wedge C \wedge \neg\psi') \vee \neg\psi''$. We get $2^{n'} \times 2^{n''}$ such formulas.

The total number of formulas in $X_\prec$ is thus $(1 + n + (n, 2) + \cdots + 1) \times 2^{n'} \times 2^{n''} = 2^n \times 2^{n'} \times 2^{n''}$ which is the number of elements of $\mathcal{C}$, as expected.

Thus, we see the kind of formulas that we find in $X_\wedge$.

Notice that in the particular case where $\mathcal{A}'$ is empty, we get the set of the formulas of the kind $A \vee B \vee C \vee \neg\psi'', A \vee B \vee \neg\psi'', A \vee \neg\psi'', \neg\psi''$, i.e. any clause which contains as disjunct a clause maximal in $\mathcal{A}''$ and which is positive in the propositions of $\mathcal{A}$. The set $(X_\prec)^\wedge = I_{f_\prec}$ of all the inaccessible formulas of $f_\prec$ is thus the set of all the formulas which are positive in $\mathcal{A}$, a well-known result.

In the particular case when $\mathcal{A}' = \mathcal{A}'' = \emptyset$, we get for $X_\prec$ the set of all the clauses positive, including the formula $\perp$. We know from property 3.4 and theorem 3.2 that $\perp$ is in $I_{f_\prec} = (X_\prec)^\wedge$ for any circumscription, because any circumscription preserves consistency, but in the other cases we do not get $\perp$ in $X_\prec$, we get $\perp$ as a conjunction of elements of $X_\prec$. In this case $(X_\prec)^\wedge = I_{f_\prec}$ is the set of all the positive formulas.

The structure of $X_\prec$ for the general circumscriptions is new to our knowledge. Precisely two things are new: 1) the description of the smallest (for $\subseteq$) set possible, in any case, and 2) the acceptation of varying predicates.

We have given a rather paradoxical way of defining the circumscriptions, using only the formulas which are inaccessible for this circumscription. We have even provided a nice duality between the preference relation associated to a circumscription (or to any cumulative preferential entailment) and the structure of some well defined subset of the set of the inaccessible formulas, the relation being $\Dashv$.

A comment is in order here, about future work on the subject. All these results are given for the finite propositional case only. It remains to see how to extend these results to traditional predicate circumscription. Some work has been done by Suchenek, in e.g. [Suc87, Suc93]. He has given a set X in the case of predicate circumscriptions ("mixed"[1], i.e. "second order circumscription in a first order framework") without varying predicates. Clearly, some restrictions are necessary, as we know that general predicate circumscription falsifies cumulativity. But it is also known that mixed predicate circumscriptions satisfy cumulativity when restricted to universal theories, and Suchenek's results concern only universal theories. Also, in predicate circumscription, it is known till [EMR85] that the equality predicate acts like a "fixed predicate", and it is effectively in this way in Suchenek's results. The great restriction of Suchenek's results concerns the fact that varying predicates are forbidden.

Now, our results could be extended to the predicate calculus (at the price of serious complications naturally). A first step would be to consider the infinite propositional calculus. However, we have found (too late to include this result here) that except if $\mathcal{A}$ is finite, or if $\mathcal{A}'$ is empty, propositional circumscriptions falsify (CR∞). This shows that our results do not extend to the case of infinite propositional circumscriptions with varying predicates.

[1]Remind that the full first order circumscription may falsify cumulativity, even without varying predicates and even restricted to universal and finitely axiomatizable theories (see example 5.2 in [MR94b]), thus this circumscription cannot be an X-mapping.

For the two particular cases $\mathcal{A}$ finite or $\mathcal{A}' = \emptyset$, a rather natural extension of our results in the finite case works, but outside these two cases, there is no hope: infinite propositional circumscriptions with varying propositions are not an X-mappings. The difficulty of extending the finite case lies in the nature of the formulas in $X_{\prec} = X_{\wedge}$ which are involved, e.g $(A \wedge B \wedge C \wedge \neg\psi') \vee \neg\psi''$. These formulas are too close to complete theories, which in the infinite propositional calculus case are not finitely axiomatizable. This seems to give strong indications that Suchenek's results do not extend fully to the case of varying predicates.

7 Conclusion and future work

We have made precise the notion of X-mapping, originating from the X-logics introduced by Siegel and Forget. Doing this, it was convenient to introduce the notion of S-mapping which seems to be of great importance in various other areas of non monotonic reasoning also. We think that further studies on S-mappings are needed.

We have given the main properties of X-mappings, in the general (i.e. infinite) case.

Then, at the price of a restriction to the finite case, we have given a transformation from any preferential entailment, or even any pre-circumscription, to X-mappings, and also conversely, in all the cases when these transformations are possible. Doing this, we have exhibited the smallest set X which is possible. We hope that this research will have some impact on the automatization of the calculus of circumscription (apparently, this was a major motivation in Siegel and Forget's work). In any case, we think that the results we have given are important from a knowledge representation side: this way of describing some preferential entailments, or even some pre-circumscriptions, only from the formulas that they cannot give as a result, is a good illustration of the power of properties such as cumulativity and case reasoning. And the duality between some of these inaccessible formulas and the corresponding preference relation sheds new lights on our comprehension of such preferential entailments.

Still in the finite case, we have also shown how to extend these results to any pre-circumscription satisfying cumulativity and the property of case reasoning.

We have recently extended these results to the infinite propositional case when this is possible. It remains to see what is the exact situation in the predicate calculus case.

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