ENCODING CLASSICAL FUSION IN ORDERED KNOWLEDGE BASES FRAMEWORK

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Abstract

The problem of merging multiple sources information is central in many information processing areas such as databases integrating problems, multiple criteria decision making, expert opinion pooling, etc. Recently, several approaches have been proposed to merge classical propositional bases, or sets of (non-prioritized) goals. These approaches are in general semantically defined. Like in belief revision, they use priorities, generally based on Dalal's distance, for merging the classical bases and return a new classical base as a result. An immediate consequence of the generation of a classical base is the impossibility of iterating the fusion process in a coherent way w.r.t. priorities since the underlying ordering is lost. This paper presents a general approach for fusing prioritized bases, both semantically and syntactically, when priorities are represented in the possibilistic logic framework. Different classes of merging operators are considered including conjunctive, disjunctive, reinforcement, adaptive and averaging operators.We show that the approaches which have been recently proposed for merging classical propositional bases can be embedded in this setting. The result is then a prioritized base, and hence the process can be coherently iterated. Moreover, we also provide a syntactic counterpart for the fusion of classical bases.

Introduction

It is well known that priorities are very important in belief revision or for fusing multiple source information. Gärdenfors (Gärdenfors 1988) has proved that any revision process which satisfies the so-called AGM postulates is based on some implicit ordering. Priorities are crucial as well to deal with conflicting sources. Even when the information provided by the sources takes the form of classical bases, (which represent non-stratified sets of pieces of knowledge or of goals without explicit priorities), several authors, e.g. (Lin 1996), (Lin and Mendelzon 1998), (Konieczny and Pino Pérez 1998, 1999), (Lin and Mendelzon 1998), (Liberatore and Schaerf 1995) and (Revescz 1993, 1997), extract implicit orderings from these classical bases.

For instance, consider the following example (Revescz 1993) where a teacher asks students which among the following languages, SQL (denoted by s), O_2 (denoted by o) and $Datalog$ (denoted by d) they would like to learn. If one provides the classical propositional base $(s \vee o) \wedge \neg d$, then the teacher assumes that the student implicitly gives two independent goals:

- learning SQL or O_2, and
- not learning $Datalog$.

This induces a set of prioritized goals where the situations $(s \wedge o \wedge \neg d)$, $(\neg s \wedge o \wedge \neg d)$ and $(s \wedge \neg o \wedge \neg d)$ are the preferred ones (since they satisfy the two sub-goals "$s \vee o$" and "$\neg d$"), while the situations $\neg s \wedge \neg o \wedge \neg d$, $s \wedge o \wedge d$, $s \wedge \neg o \wedge d$, $\neg s \wedge o \wedge d$ are less preferred (since they only satisfy one sub-goal), and lastly the situation $\neg s \wedge \neg o \wedge d$ is the least preferred one since both sub-goals are falsified. These priorities can be obtained using Dalal's distance (Dalal 1988).

The fusion of information expressed in a logical form has raised an increasing interest in the recent past years (Baral et al. 1992, Cholvy 1998, Konieczny and Pino Pérez 1998, 1999, Lin 1996, Lin and Mendelzon 1998). Indeed this problem naturally occurs when handling multiple sources of information, and trying to extract the common, conflict-free part of the information, or when trying to fuse the goals expressed by several agents. In this paper we propose a general approach for fusing prioritized propositional bases in the framework of possibilistic logic. Possibilistic logic, e.g. (Dubois et al. 1994), offers a framework for reasoning with classical logic formulas associated with weights belonging to a lattice structure in the most general case, and often in practice to a totally ordered scale, such as $[0, 1]$, or a subset of $[0, 1]$.

Prioritized propositional bases are sets of weighted formulas of the form $B = \{(\phi_i, \alpha_i) : i = 1, n\}$ where ϕ_i is a classical formula, and α_i belongs to $[0, 1]$ encoding the level of priority of ϕ_i. Each possibilistic base B induces a possibility distribution π_B, which rank-orders the different interpretations. We show that this general framework allows us to recover classical fusion methods recently proposed in (Konieczny and Pino Pérez 1998, 1999, Lin 1996, Lin and Mendelzon 1998). This is obtained by the syntactic association of a set of weighted

formulas to each classical base, and by providing the possibilistic counterpart of any classical merging operator. We show that this process can be iterated in a coherent way, and also provides a syntactic way for computing classical fusion operators which are only semantically defined generally.
The next section recalls the fusion of classical propositional bases. Then we provide the necessary background on possibilistic logic. General classes of combination operators are introduced and studied in the main part of the paper. Lastly, we show how to encode classical fusion in the possibilistic logic framework.

Fusion of classical propositional bases

We consider a propositional language $\mathcal{L}$ over a finite alphabet $\mathcal{P}$ of atoms. A literal is an atom or a negation of an atom. Ω denotes the set of all interpretations. Logical equivalence is denoted by $\equiv$. Classical disjunction and conjunction are respectively represented by $\vee, \wedge$. Let ψ be a formula of $\mathcal{L}$, $[\psi]$ denotes the set of all models of ψ. K denotes a classical propositional base. Let $E = \{K_1, ..., K_n\}$ $(n \geq 1)$ be a multi-set of n consistent propositional bases to be merged. E is called an information set. We define a merging or combination operator Δ as a function which associates to each information set E a classical propositional base, denoted by $\Delta(E)$.
In the literature, we can distinguish two different ways for defining $\Delta(E)$: either we use some implicit priority or not. This section discusses the first case in details, but let us first summarize how to build $\Delta(E)$ when no implicit priority is assumed.
There are two straighforward ways for defining $\Delta(E)$, depending if the sources are conflicting or not, namely:

- classical conjunctive merging:
$$\Delta(E) = \bigwedge_{K_i \in E} K_i.$$
- classical disjunctive merging:
$$\Delta(E) = \bigvee_{K_i \in E} K_i.$$

Clearly the first one is only appropriate when the bases are not conflicting, while the second has the opposite behaviour. However if disjunctive merging operators always guarantee the consistency of $\Delta(E)$ (if all the K_i's are consistent), they are very cautious. To avoid this drawback, more parcimonious approaches have been developed (Rescher and Manor 1970) when $\wedge_{K_i \in E} K_i$ is inconsistent. They are based on the notion of maximal consistent subbases of E, which are maximal either w.r.t. the inclusion criterion or w.r.t. cardinality criterion. Once these subsets are computed, $\Delta(E)$ is defined as the intersection of the deductive closure of these subsets. This is clearly in between the above conjunctive and disjunctive merging attitudes. Moreover, this may be still found very cautious, and a way for avoiding this problem is the introduction of priorities.
We now present in details some recent approaches proposed for merging classical bases which use implicit priority.

Basic steps and local distances

The three basic steps followed in (Konieczny and Pino Pérez 1998, 1999, Lin 1996, Lin and Mendelzon 1998) for the semantic merging of an information set E by an operator Δ are:

1. Rank-order the set of interpretations Ω with respect to each propositional base K_i by computing a local distance, denoted by $d(\omega, K_i)$, between ω and each K_i.
2. Rank-order the set of interpretations Ω w.r.t. all the propositional bases. This leads to the overall distance denoted by $d_\Delta(\omega, E)$. This latter, computed from local distances $d(\omega, K_i)$, defines an ordering relation between the interpretations defined as follows:
$$\omega \leq_E^\Delta \omega' \text{ iff } d_\Delta(\omega, E) \leq d_\Delta(\omega', E).$$
3. $\Delta(E)$, the result of the merging process, is defined by being such that its models are minimal with respect to $\leq_E^\Delta$, namely
$$[\Delta(E)] = min(\Omega, \leq_E^\Delta).$$

In the works above-mentioned, the local distance $d(\omega, K_i)$ is based on Dalal's distance (Dalal 1988). The distance between an interpretation ω and a propositional base K_i is defined by the least number of atoms on which this interpretation differs from some model of the propositional base. More formally,
$$d(\omega, K_i) = min_{\omega' \in [K_i]} dist(\omega, \omega'),$$
where $dist(\omega, \omega')$ is the number of atoms whose valuations differ in the two interpretations.

Example 1 *Let us extend the example of the introduction with three students. The first student wants to only learn SQL or* O_2*:* $K_1 = (s \vee o) \wedge \neg d$*. The second wants to only learn Datalog or* O_2 *but not both:* $K_2 = (\neg s \wedge d \wedge \neg o) \vee (\neg s \wedge \neg d \wedge o)$*. The third wants to learn the three languages:* $K_3 = (s \wedge d \wedge o)$*. Let* $\omega_0 = \neg s \neg d \neg o$, $\omega_1 = \neg s \neg d o$, $\omega_2 = \neg s d \neg o$, $\omega_3 = \neg s d o$, $\omega_4 = s \neg d \neg o$, $\omega_5 = s \neg d o$, $\omega_6 = s d \neg o$, $\omega_7 = s d o$*.*
We give in the following table Dalal's distances between each interpretation and the base:

ω	$d(\omega, K_1)$	$d(\omega, K_2)$	$d(\omega, K_3)$
ω_0	1	1	3
ω_1	0	0	2
ω_2	2	0	2
ω_3	1	1	1
ω_4	0	2	2
ω_5	0	1	1
ω_6	1	1	1
ω_7	1	2	0

Table 1

Aggregating local distances

Once $d(\omega, K_i)$ is defined for each K_i, several methods have been proposed in order to aggregate the local distances $d(\omega, K_i)$ according to whether the bases have the same importance or not. In particular the following operators have been proposed:

- Majority operator (Lin and Mendelzon 1998, Revescz 1993):
$$d_{\Sigma}(\omega, E) = \Sigma_{i=1}^{n} d(\omega, K_i).$$
- Weighted sum operator (Lin 1996):
$$d_{ws}(\omega, E) = \Sigma_{i=1}^{n} d(\omega, K_i) * k_i,$$
where k_i are integer numbers reflecting the level of importance of the propositional bases.
- Max-based egalitarist operator (Revescz 1997):
$$d_{max}(\omega, E) = max_{i=1,n} d(\omega, K_i).$$
- Lexmin-based (or generalized max) egalitarist operator (Konieczny and Pino Pérez 1998,1999):
this aggregating operator compares vectors of distances. Let $d_j^{\omega} = d(\omega, K_j)$. Denote $d_{gmax}(\omega, E)$ the result of sorting the vector $(d_1^{\omega} ... d_n^{\omega})$ in a decreasing way.
Then, $\omega \in [\Delta(E)]$ if there is no ω' such that $d_{gmax}(\omega', E) <_{Lex} d_{gmax}(\omega, E)$,
where $<_{Lex}$ is defined as follows:
Let $S_1 = (s_1 ... s_n)$ and $S_2 = (s'_1 ... s'_n)$ be two sequences of integers given in a decreasing order. Then, $S_1 <_{Lex} S_2$ if and only if $\exists k \leq n$, such that $s_k < s'_k$ and $\forall i < k : s_i = s'_i$.
Moreover $S_1 =_{Lex} S_2$ if $\forall k \leq n$, $s_k = s'_k$.

Example 1 (continued)
Let $k_1 = k_3 = 1, k_2 = 3$ for ws operator.

ω	Σ	ws	max	$gmax$
ω_0	5	7	3	$(3,1,1)$
ω_1	**2**	**2**	2	$(2,0,0)$
ω_2	4	4	2	$(2,2,0)$
ω_3	3	5	**1**	$(1,1,1)$
ω_4	4	8	2	$(2,2,0)$
ω_5	**2**	4	**1**	**(1,1,0)**
ω_6	3	5	**1**	$(1,1,1)$
ω_7	3	7	2	$(2,1,0)$

Table 2 (Bold elements are the preferred interpretations)

Then we get: $[\Delta_{\Sigma}(E)] = \{\omega_1, \omega_5\}$, $[\Delta_{ws}(E)] = \{\omega_1\}$, $[\Delta_{max}(E)] = \{\omega_3, \omega_5, \omega_6\}$ and $[\Delta_{gmax}(E)] = \{\omega_5\}$.

As it can be seen in this example, max is the most cautious one, and refined by $gmax$. Besides ws and Σ which correspond to other points of views, may select other interpretations.

Non-iteration (loss of associativity) of the process

The main drawback of Δ is the non-iteration of the process in a coherent way w.r.t. Dalal's distance even for associative operators. Indeed, in general $d(\omega, \Delta_f(K_1, K_2)) \neq f(d(\omega, K_1), d(\omega, K_2))$, where Δ_f is a merging operator based on an associative function f. This is due to the fact that when computing $d(\omega, \Delta_f(K_1, K_2))$ we lost the underlying priority between K_1 and K_2.
Let us now consider the above example and the majority operator which is associative.
Let $K' = \Delta_{\Sigma}(\{K_1, K_2\})$. We can easily check that $d(\omega, \Delta_{\Sigma}(\{K_1, K_2\})) \neq d(\omega, K_1) + d(\omega, K_2)$. Therefore, it is not surprising to find $\Delta_{\Sigma}(\{K', K_3\}) \not\equiv \Delta_{\Sigma}(\{K_1, K_2, K_3\})$. Indeed, we have $[\Delta_{\Sigma}(\{K', K_3\})] = \{\omega_1, \omega_3, \omega_5, \omega_7\}$, while $[\Delta_{\Sigma}(\{K_1, K_2, K_3\})] = \{\omega_1, \omega_5\}$.

Background on possibilistic logic and fusion

This section recalls some basic notions of possibilistic logic e.g., (Dubois et al. 1994).

Possibility distributions

At the semantic level, possibilistic logic is based on the notion of a possibility distribution, denoted by π, which is a mapping from Ω to [0,1] representing the available information. $\pi(\omega)$ represents the degree of compatibility of the interpretation ω with the available beliefs about the real world if we are representing uncertain pieces of knowledge (or the degree of satisfaction of reaching state ω if we are modelling preferences). By convention, $\pi(\omega) = 1$ means that it is totally possible for ω to be the real world (or that ω is fully satisfactory), $1 > \pi(\omega) > 0$ means that ω is only somewhat possible (or satisfactory), while $\pi(\omega) = 0$ means that ω is certainly not the real world (or not satisfactory at all). Associated with a possibility distribution π is the necessity degree of any formula ϕ: $N(\phi) = 1 - \Pi(\neg\phi)$ which evaluates to what extent ϕ is entailed by the available beliefs, and defined from the consistency degree of a formula ϕ w.r.t. the available information, $\Pi(\phi) = max\{\pi(\omega) : \omega \in [\phi]\}$, where $[\phi]$ denotes the set of all models of ϕ.
Note the mapping N reverses the scale on which π is ranging, and that $N(\phi) = 1$ means that ϕ is a totally certain piece of knowledge or a compulsory goal, while $N(\phi) = 0$ expresses the complete lack of knowledge or of priority about ϕ, but does not mean that ϕ is or should be false.
In the rest of the paper, $a, b, c, ...$ reflect the possibility degrees of the interpretations.

Possibilistic logic bases

At the syntactic level, uncertain pieces of information are represented by means of *a possibilistic knowledge base* which is a set of weighted formulas $B = \{(\phi_i, \alpha_i) : i = 1, n\}$ where ϕ_i is a classical formula and α_i belongs to a totally ordered scale such as [0,1]. (ϕ_i, α_i) means that the certainty degree of ϕ_i is at least equal to α_i

$(N(\phi_i) \geq \alpha_i)$.We denote by B^* the classical base associated with B, namely the base obtained from B by forgetting the weights of formulas. A possibilistic base B is consistent if and only if its classical base B^* is consistent.
In the following, $\alpha, \beta, \gamma, \ldots$ reflect the weights associated with formulas.
Given B, we can generate a unique possibility distribution, denoted by π_B, such that all the interpretations satisfying all the beliefs in B will have the highest possibility degree, namely 1, and the other interpretations will be ranked w.r.t. the highest belief that they falsify, namely we get (Dubois et al. 1994):

Definition 1 *$\forall \omega \in \Omega$,*

$$\pi_B(\omega) = \begin{cases} 1 & \forall (\phi_i, \alpha_i) \in B, \omega \in [\phi_i] \\ 1 - max\{\alpha_i : \omega \notin [\phi_i]\} & otherwise. \end{cases}$$

We now give further definitions which are used in the paper:

Definition 2 *Let B be a possibilistic knowledge base, and $\alpha \in [0,1]$. We call the α-cut (resp. strict α-cut) of B, denoted by $B_{\geq\alpha}$ (resp. $B_{>\alpha}$), the set of classical formulas in B having a certainty degree at least equal to α (resp. strictly greater than α).*

Definition 3 *B and B' are said to be equivalent, denoted by $B \equiv_s B'$, iff $\forall \alpha \in [0,1]$, $B_{\geq\alpha} \equiv B'_{\geq\alpha}$, where $\equiv$ is the classical equivalence.*

The equivalence is understood in the sense that the two possibilistic bases generate the same possibility distributions.

$Inc(B) = max\{\alpha_i : B_{\geq\alpha_i} \textit{ is inconsistent}\}$ denotes the inconsistency degree of B.
When B is consistent, we have $Inc(B) = 0$.

Subsumption can now be defined:

Definition 4 *Let (ϕ, α) be a belief in B. Then, (ϕ, α) is said to be subsumed by B if:*

$$(B - \{(\phi, \alpha)\})_{\geq\alpha} \vdash \phi.$$

(ϕ, α) is said to be strictly subsumed by B if $B_{>\alpha} \vdash \phi$.

Then we have the following result:

Lemma 1 *Let (ϕ, α) be a subsumed information in B. Then B and $B' = B - \{(\phi, \alpha)\}$ are equivalent.*

Lastly, weights are propagated out in the inference process in the following way:

Definition 5 *Let B be a possibilistic base. Let (ϕ, α) be a piece of information with $\alpha > Inc(B)$.*
(ϕ, α) is said to be a consequence of B, denoted by $B \vdash_\pi (\phi, \alpha)$, iff $B_{\geq\alpha} \vdash \phi$.

Hence, the useful consistent part of B, for making possibilistic inference, is made of the formulas whose weights are above the inconsistency level, namely
$\rho(B) = \{\phi_i : (\phi_i, \alpha_i) \in B \textit{ and } \alpha_i > Inc(B)\}$.
Indeed, we can check that:
$\rho(B) \vdash \psi$ iff $\exists \alpha > Inc(B), B \vdash_\pi (\psi, \alpha)$.

Fusion of prioritized knowledge bases

We first recall a general result underlying the fusion process in possibilistic logic (Benferhat et al. 1999).
Let B_1, B_2 be two possibilistic bases, and π_1 and π_2 be their associated possibility distributions. Let $\oplus$ be a two place function whose domain is $[0,1] \times [0,1]$ (to be used for aggregating $\pi_1(\omega)$ and $\pi_2(\omega)$). The only requirements for $\oplus$ are the following properties:
i. $1 \oplus 1 = 1$,
ii. If $a \geq c$, $b \geq d$ then $a \oplus b \geq c \oplus d$ (monotonicity).
The first one acknowledges the fact that if two sources agree that ω is fully possible (or satisfactory), then the result of the combination should confirm it. The second property expresses that a degree resulting from a combination cannot decrease if the combined degrees increase.
In (Benferhat et al. 1999), it has been shown that the syntactic counterpart of the fusion of π_1 and π_2 is the following possibilistic base, denoted by $\mathcal{B}_\oplus$ (and sometimes by $B_1 \oplus B_2$), and which is made of the union of:

- the initial bases with new weights defined by:

$$\{(\phi_i, 1-(1-\alpha_i)\oplus 1) : (\phi_i, \alpha_i) \in B_1\} \cup \{(\psi_j, 1-1\oplus(1-\beta_j)) : (\psi_j, \beta_j) \in B_2\} \quad (1)$$

- and the knowledge common to B_1 and B_2 defined by:

$$\{(\phi_i \vee \psi_j, 1-(1-\alpha_i)\oplus(1-\beta_j)) : (\phi_i, \alpha_i) \in B_1 \textit{ and } (\psi_j, \beta_j) \in B_2\}$$

It has been shown that $\pi_{\mathcal{B}_\oplus}(\omega) = \pi_1(\omega) \oplus \pi_2(\omega)$ where $\pi_{\mathcal{B}_\oplus}$ is the possibility distribution associated to $\mathcal{B}_\oplus$ using Definition 1.

In the case of n sources, the syntactic computation of the resulting base can be easily applied when $\oplus$ is associative. The syntactic generalization for non-associative operator ($\oplus$ is then a n-ary operator defined on vector of possibility distributions) is also possible.

The next section analyses several classes of $\oplus$ which cope with different issues met in merging multiple sources information.

General classes of possibilistic merging operators

Conjunctive operators

One of the important aims in merging uncertain information is to exploit complementarities between the sources in order to get a more complete and precise global point of view. Since we deal with prioritized information, two kinds of complementarities can be considered depending on whether we refer to formulas only, or to priorities attached to formulas. In this subsection, we introduce conjunctive operators which exploit the symbolic complementarities between sources. More formally,

Definition 6 *$\oplus$ is said to be a conjunctive operator if* $\forall a \in [0,1], a \oplus 1 = 1 \oplus a = a$.

The following proposition shows indeed that conjunctive operators, in case of consistent sources of information, exploit their complementarities by recovering all the symbolic information.

Proposition 1 *Let B_1 and B_2 be such that $B_1^* \wedge B_2^*$ is consistent. Let $\oplus$ be a conjunctive operator. Then,*

$$\mathcal{B}_{\oplus}^* \equiv B_1^* \wedge B_2^*.$$

The proof can be easily checked by noticing that the general writing of $\mathcal{B}_{\oplus}$ for conjunctive operators is:

$$B_1 \cup B_2 \cup \{(\phi_i \vee \psi_j, 1-(1-\alpha_i) \oplus (1-\beta_j))\}$$

obtained by replacing in (1), $(1-\alpha_i) \oplus 1$ and $1 \oplus (1-\beta_j)$ by $1-\alpha_i$ and $1-\beta_j$ respectively.

An important feature of a conjunctive operator is its ability to give preference to more specific information. Namely, if a source of information S_1 contains all the information provided by S_2, then combining S_1 and S_2 with a conjunctive operator leads simply to S_1. More formally,

Proposition 2 *Let B_1 and B_2 be such that $\forall(\psi, \beta) \in B_2, B_1 \vdash_\pi (\psi, \beta)$. Then,*

$$\mathcal{B}_{\oplus}^* \equiv B_1^*.$$

An example of a conjunctive operator is the *minimum* (for short *min*), for which we can easily check that $\mathcal{B}_{\oplus} = B_1 \cup B_2$. Indeed if $\oplus = min$, the formula in the common knowledge $(\phi_i \vee \psi_j, 1-(1-\alpha_i) \oplus (1-\beta_j))$ is subsumed either by (ϕ_i, α_i) if $a_i \geq \beta_j$ or by (ψ_j, β_j) otherwise.
Other examples are the product, and the geometric average defined by $a \oplus b = \sqrt{ab}$.

Disjunctive operators

Another important issue in information fusion is how to deal with conflicts. When all the sources are equally reliable and conflicting, we should not introduce any preference between the sources, and we should not infer from the result of the combination all the information provided by one of the sources. Namely, if $B_1 \cup B_2$ is inconsistent, then one can require that $\mathcal{B}_{\oplus}$ neither infer B_1 nor B_2. Such a behaviour cannot be captured by any conjunctive operators. This requirement is captured by the disjunctive operators defined by:

Definition 7 *$\oplus$ is said to be a disjunctive operator if* $\forall a \in [0,1], a \oplus 1 = 1 \oplus a = 1$.

Then, we have:

Proposition 3 *Let B_1 and B_2 be such that $B_1^* \wedge B_2^*$ is inconsistent. Then, there exists $(\phi, \alpha) \in B_1$ and $(\psi, \beta) \in B_2$ such that $\mathcal{B}_{\oplus} \nvdash_\pi (\phi, \alpha)$ and $\mathcal{B}_{\oplus} \nvdash_\pi (\psi, \beta)$.*

Note that if $\oplus$ is a disjunctive operator then $\mathcal{B}_{\oplus}$ is of the form:

$$\mathcal{B}_{\oplus} = \{(\phi_i \vee \psi_j, 1-(1-\alpha_i) \oplus (1-\beta_j))\}.$$

Now, a second natural requirement that one may ask for, in case of conflicts, is to recover the disjunction of all the symbolic information provided by the sources. Clearly, it is easy to find a disjunctive operator which does not satisfy this second requirement. A trivial case is to take the "vacuous" disjunctive operator defined by: $\forall a, \forall b, a \oplus b = 1$.
To satisfy this second requirement we define the notion of regular disjunctive operator:

Definition 8 *An operator $\oplus$ is said to be a regular disjunctive operator if:*

- $\forall a, a \oplus 1 = 1 \oplus a = 1$
- $\forall a \neq 1, \forall b \neq 1, a \oplus b \neq 1$

Then, we have:

Proposition 4 *Let B_1 and B_2 be two bases and $\oplus$ be a regular disjunctive operator. Then,*

$$\mathcal{B}_{\oplus}^* \equiv B_1^* \vee B_2^*.$$

Examples of regular disjunctive operators are the *max*, the so-called *"probabilistic sum"* defined by: $a \oplus b = a + b - ab$, and the dual of the geometric average defined by $a \oplus b = 1 - \sqrt{(1-a)(1-b)}$.
Lastly, note that regular disjunctive operators are not appropriate in the case of consistent sources; in particular they give preference to less specific information. Namely, if B_1 and B_2 are two bases such that $\forall(\psi, \beta) \in B_2, B_1 \vdash_\pi (\psi, \beta)$ then we have $\mathcal{B}_{\oplus} \equiv_s B_2$.

Idempotent operators

Another important problem in fusing multiple sources information is how to deal with redundant information. There are two different situations: either we ignore the redundancies, which is suitable when the sources are not independent, or we view redundancy as a confirmation of the same information provided by independent sources. Idempotent operations are defined by:

Definition 9 *$\oplus$ is said to be an idempotent operator if* $\forall a \in [0,1], a \oplus a = a$.

Idempotent operators aim to ignore direct redundancies. Namely, if two sources of information entail the same formula ϕ to a degree α, then one may require that the fused base should not entail ϕ with a degree higher than α. However, such a requirement is strong since ϕ can be obtained from another path exploiting complementarities between higher level formulas provided by the two sources. This is illustrated by the following example:

Example 2 *Let* $B_1 = \{(\psi, .9); (\phi, .2)\}$ *and*

$$B_2 = \{(\phi \vee \neg\psi, .8); (\phi, .2)\}.$$

Clearly $B_1 \vdash_\pi (\phi, .2)$ and $B_2 \vdash_\pi (\phi, .2)$. Now let $\oplus$ an idempotent operator defined by: $a \oplus b = \frac{a+b}{2}$.

Then $\mathcal{B}_{\oplus}= \{(\psi,.45);(\phi,.1)\} \cup \{(\phi\vee\neg\psi,.4);\ (\phi,.1)\}\cup$
$\{(\phi\vee\psi,.55);(\phi\vee\neg\psi,.5);(\phi,.2)\}$
$\equiv_s \{(\psi,.45);(\phi\vee\psi,.55);(\phi\vee\neg\psi,.5)\}$.
We can easily check that $\mathcal{B}_{\oplus}\vdash_\pi (\phi,.5)$ *with* $.5 \geq .2$. *This is mainly due to the two pieces of information* $(\psi,.9)$ *and* $(\phi\vee\neg\psi,.8)$*, provided separately by the sources.*

Now, the following proposition shows the cases where idempotent operators indeed ignore redundancies:

Proposition 5 *Let* B_1 *and* B_2 *be two bases, and* $\oplus$ *be an idempotent operator. Let* ϕ *be such that* $B_1 \vdash_\pi (\phi,\alpha)$*;* $B_2 \vdash_\pi (\phi,\beta)$ *with* $\beta\leq\alpha$. *Let* $\Gamma = B_{1>\alpha}\cup B_{2>\alpha}$. *Then,*
if $\Gamma \not\vdash \phi$ *then* $\mathcal{B}_{\oplus}\vdash_\pi (\phi,\gamma)$*, with* $\gamma \leq max(\alpha,\beta)$.

Note that γ may be equal to 0 in case of inconsistency.

Γ in this proposition is the set of classical formulas in B_1 and B_2 having a weight strictly greater than α. If ϕ cannot be deduced from Γ then the idempotent property only guarantees that the repeated information will not be inferred with a priority higher than the one with which it can be individually obtained from the different sources.

The proof of the above proposition can be checked by noticing that any idempotent operator is between the *minimum* and the *maximum*. Indeed, let $\oplus$ be an idempotent operator. Then,
$a\oplus b \geq a\oplus a$ if $b\geq a$
$\Rightarrow a\oplus b\geq a$
$\Rightarrow a\oplus b \geq min(a,b)$.

Moreover: $b\oplus b\geq a\oplus b$ when $b\geq a$
$\Rightarrow b\geq a\oplus b$
$\Rightarrow max(a,b)\geq a\oplus b$.

From this fact, combining B_1 and B_2 leads to:
$\{(\phi_i, 1-(1-\alpha_i)\oplus 1)\} \cup \{(\psi_j, 1-1\oplus(1-\beta_j))\}$ $\cup\{(\phi_i\vee\psi_j, 1-(1-\alpha_i)\oplus(1-\beta_j))\}$.
Now, $1-(1-\alpha_i)\oplus 1\leq \alpha_i$, $1-1\oplus(1-\beta_j)\leq\beta_j$ and $1-(1-\alpha_i)\oplus(1-\beta_j)\leq max(\alpha_i,\beta_j)$.
Therefore, $\mathcal{B}_{\oplus>\alpha}$ is at most equivalent to Γ (It can contain less formulas than Γ).

Reinforcement operators

Reinforcement operators in case of repeated information have a confirmation effect. Namely, if the same piece of information is supported by two different sources, then the priority attached to this piece of information should be strictly greater than the one provided by the sources. A first formal class of reinforcement operators can be defined as follows:

Definition 10 $\oplus$ *is said to be a reinforcement operator if* $\forall a,b\neq 1$ *and* $a,b\neq 0, a\oplus b < min(a,b)$.

We can easily check that if we aggregate the two pieces of information (ϕ,α) and (ϕ,β), then the resulting base is: $\{(\phi, f(\alpha,\beta))\}$ where $f(\alpha,\beta) = 1-(1-\alpha)\oplus(1-\beta) > max(\alpha,\beta)$ for $\alpha,\beta\in(0,1)$.

Besides, one can require that reinforcement operators recover all the common information with a higher weight. Namely if the same formula is a plausible consequence of each base, then this formula should be accepted in the fused base with a higher priority. The following proposition shows a first case where this result holds:

Proposition 6 *Let* B_1 *and* B_2 *be such that* $B_1^*\wedge B_2^*$ *is consistent. Let* ϕ *be such that* $B_1\vdash_\pi(\phi,\alpha)$ *and* $B_2\vdash_\pi(\phi,\beta)$ *where* α *and* β *are strictly positive. Then,*
$$\mathcal{B}_{\oplus}\vdash_\pi(\phi,\gamma),$$
with $\gamma > max(\alpha,\beta)$ *if* $\alpha,\beta\in(0,1)$*, and* $\gamma=1$ *if* $\alpha=1$ *or* $\beta=1$.

Now, in case of conflicts, and more precisely, in case of a strong conflict, namely $Inc(B_1\cup B_2)=1$, then the above proposition does not hold.

Example 3 *Let* $B_1=\{(\phi,1),(\psi,\alpha)\}$ *and*
$B_2=\{(\neg\phi,1),(\psi,\beta)\}$.
We have $\mathcal{B}_{\oplus}=\{(\phi,1);(\psi,1-(1-\alpha)\oplus 1)\}\cup\{(\neg\phi,1);(\psi,1-1\oplus(1-\beta))\}\cup\{(\phi\vee\psi,1);(\psi\vee\neg\phi,1);\ (\psi,1-(1-\alpha)\oplus(1-\beta))\}$.
We have $Inc(\mathcal{B}_{\oplus})=1$*, then we cannot infer* ψ *from* $\mathcal{B}_{\oplus}$.

In the above example, since $Inc(B_1\cup B_2)=1$ then even if we add $(\psi,1)$ to $B_1\cup B_2$ explicitly then ψ cannot be recovered. In possibilistic logic, when there is a strong conflict, such as conflicts between integrity constraints, then only tautologies are plausible consequences. In this case it is better to use a regular disjunctive operator.
So the first condition is to avoid that $Inc(B_1\cup B_2)=1$. But this is not enough since even if $Inc(B_1\cup B_2)<1$ one can have $Inc(\mathcal{B}_{\oplus})=1$ due to the reinforcement effect which can push the priority of conflicting information to the maximal priority allowed, namely 1. For instance, let us consider the excessively optimistic reinforcement operator defined by:

$$\forall a,\forall b, a\neq 1, b\neq 1, a\oplus b = b\oplus a = 0.$$

Then we can check that as soon as there is a conflict between the bases to be merged, the inconsistency degree of the fused base will reach the maximal value.
The following definition focuses on a more interesting class of reinforcement operations:

Definition 11 *The reinforcement operation* $\oplus$ *is said to be progressive if*
$$\forall a,b\neq 0, a\oplus b\neq 0.$$

The progressive operation guarantees that if a some formula (ϕ,α) with $\alpha>0$ is inferred by the sources then this formula belongs to $\mathcal{B}_\oplus$ with a weight β such that $\alpha < \beta < 1$. However, this new weight β can be less than the inconsistency degree of $\mathcal{B}_\oplus$ and therefore ϕ will be drowned by the inconsistency of the database. This situation is illustrated by the following example:

Example 4
Let $B_1 = \{(\phi \vee \psi, .9); (\phi, .5); (\psi, .5); (\xi, .1)\}$ *and* $B_2 = \{(\neg\phi \vee \neg\psi, .9); (\neg\phi, .5); (\neg\psi, .5); (\xi, .1)\}$.
Clearly, each base entails ξ *which is largely below the inconsistency degree of* $B_1 \cup B_2$. *Now, let us compute* $B_1 \oplus B_2$ *with the product operator which is a progressive operator. We get:*
$B_1 \oplus B_2 = B_1 \cup B_2 \cup \{(\phi \vee \psi \vee \xi, .91); (\neg\phi \vee \neg\psi \vee \xi, .91); (\phi \vee \neg\psi, .75); (\neg\phi \vee \psi, .75); (\phi \vee \xi, .55); (\psi \vee \xi, .55); (\neg\phi \vee \xi, .55); (\neg\psi \vee \xi, .55); (\xi, .19)\}$.
Note that there is a reinforcement on ξ *since its new weight is* $.55$ *(which can be obtained for instance from* $(\phi \vee \xi, .55)$ *and* $(\neg\phi \vee \xi, .55)$*). However, this new weight is less than the inconsistency degree of* $\mathcal{B}_\oplus$ *which is of* $.75$*, higher than* $Inc(B_1 \cup B_2) = .5$.

The following proposition generalizes Proposition 6 and shows that if the inconsistency degree does not increase, then the common knowledge is entailed.

Proposition 7 *Let* B_1 *and* B_2 *be such that* $Inc(B_1 \cup B_2) \neq 1$. *Let* ϕ *be such that* $B_1 \vdash_\pi (\phi, \alpha)$ *and* $B_2 \vdash_\pi (\phi, \beta)$ *with* $\alpha > 0, \beta > 0$. *Let* $\oplus$ *be a progressive reinforcement operation. Then,*
if $Inc(\mathcal{B}_\oplus) = Inc(B_1 \cup B_2)$ *then,* $\mathcal{B}_\oplus \vdash_\pi (\phi, \gamma)$,
with $\gamma > max(\alpha, \beta)$, *and* $\gamma = 1$ *if* $\alpha = 1$ *or* $\beta = 1$.

The following proposition shed light on the nature of the reinforcement operator:

Proposition 8 *Let* B_1 *be a possibilistic base, and* B_2 *another possibilistic base which is not conflicting with completely certain formulas of* B_1.
Let $\oplus$ *be a progressive operator. Denote by* B_2^n *the combination of* B_2 n *times with* $\oplus$. *Then, there exists* n *such that:*

$$B_1 \oplus B_2^n \vdash_\pi B_2.$$

This proposition means that reinforcement operators are majority operators, in the sense that if the same piece of information is repeated enough time then this piece of information will be believed.

Adaptive merging operators

The regular disjunctive operators appear to be appropriate when the sources are completely conflicting. However, in the case of consistency, or of a low level of inconsistency, regular disjunctive operators are very cautious. Besides, reinforcement is not appropriate in the case of complete conflicts.
The aim of adaptive operators is to have a disjunctive behaviour in case of complete contradiction and the progressive reinforcement behaviour in the other case.
Let $\oplus_d$ and $\oplus_r$ be respectively a regular disjunctive and a progressive reinforcement operator. Let h be either equal to 1 or to 0. Then we define an adaptive operation, denoted by $\oplus_h$, as follows:

$$a \oplus_h b = max(min(h, (a \oplus_d b)), min(1 - h, (a \oplus_r b)))$$

Then we have the following result:

Proposition 9 *Let* B_1 *and* B_2 *be two possibilistic bases. Let* h *be equal to* 1 *if* $Inc(B_1 \cup B_2) = 1$ *and equal to* 0 *otherwise. Let* $\oplus_h$ *be an adaptive operator. If* $Inc(\mathcal{B}_\oplus) = Inc(B_1 \cup B_2)$ *then,* $\forall\phi$, *if* $B_1 \vdash_\pi (\phi, \alpha)$ *and* $B_2 \vdash_\pi (\phi, \beta)$ *then we have:* $\mathcal{B}_{\oplus_h} \vdash (\phi, \gamma)$ *with* $\gamma > 0$.

Averaging operators

Another class of combination operators of interest are *averaging operators*, well known for aggregating preferences. They are defined as follows:

Definition 12 $\oplus$ *is called an averaging operator if*

$$max(a, b) \geq a \oplus b \geq min(a, b)$$

(and $\oplus \neq max$, $\oplus \neq min$*).*

One example of averaging operators is the arithmetic mean $a \oplus b = \frac{a+b}{2}$. In this case, at the syntactic level, the result of combining B_1 and B_2 writes:

$$\{(\phi_i, \tfrac{\alpha_i}{2})\} \cup \{(\psi_j, \tfrac{\beta_j}{2})\} \cup \{(\phi_i \vee \psi_j, \tfrac{\alpha_i + \beta_j}{2})\}.$$

From this writing, in case of consistency we can check that if $B_1 \vdash_\pi (\phi, \alpha)$ and $B_2 \vdash_\pi (\phi, \beta)$ then $\mathcal{B}_\oplus \vdash_\pi (\phi, \gamma)$ with $\gamma \geq \frac{\alpha+\beta}{2}$.

Fusion of a classical bases encoded in possibilistic logic

In previous sections,we have presented merging operators for both classical and prioritized bases. In this section, we show that possibilistic logic can recover the classical merging approaches reviewed in the previous sections. Let us first show how to associate a possibilistic base to a given classical base. Technically speaking our aim in this paper is, given a multi-set of classical propositional bases E, and a classical merging operator Δ, to show how to compute the set of models of the resulting base using possibilistic merging operators. To reach this aim, we first associate to each K_i in E a prioritized base B_{K_i}. Roughly speaking, this base should be such that $\pi_{B_{K_i}}$, the possibility distribution associated to B_{K_i}, be a function of the Dalal distance associated to K_i. Let $\mathcal{B} = \{B_{K_1}, ..., B_{K_n}\}$ be the prioritized counterpart of $E = \{K_1, ..., K_n\}$. Then for each Δ applied to E and yielding the knowledge base $\Delta(E)$, we define a possibilistic aggregation operator $\oplus$ applied to $\mathcal{B} = \{B_{K_1}, ..., B_{K_n}\}$ and yielding a prioritized base $\mathcal{B}_\oplus$. This base should be such that the set of preferred models in $\pi_{\mathcal{B}_\oplus}$, denoted $[\mathcal{B}_\oplus]$, is equal to $[\Delta(E)]$ the set of models of $\Delta(E)$, as pictured on Figure 1.

$$\begin{array}{ccc} E=\{K_1,...,K_n\} & \longrightarrow & \mathcal{B}=\{B_{K_1},...,B_{K_n}\} \\ \downarrow\Delta & & \downarrow\oplus \\ d_\Delta(\omega,E) & & \mathcal{B}_\oplus \text{ s.t. } \pi_{\mathcal{B}_\oplus}(\omega)=f(d_\Delta(\omega,E)) \\ \searrow & & \swarrow \\ [\Delta(E)] & = & max_{\omega\in\Omega}\pi_{\mathcal{B}_\oplus} \end{array}$$

Figure 1

From a classical base to a possibilistic base

The aim of this subsection is given a classical knowledge base K we construct a possibilistic base B_{K_i} such that:

$$\forall\omega, \pi_{B_{K_i}}(\omega)=f(d(\omega,K_i)).$$

First, we consider a base K which is a set of *literals* $K=\{l_i : i\in I\}$, namely K contains one formula composed of a conjunction of literals. To illustrate the construction of the possibilistic base associated to K, we use the base $K_3=\{s\wedge d\wedge o\}$ of our example. Let us show how to construct B_{K_3}. Note that using Dalal distance, the worst interpretations are the ones which falsify all literals in K_3. At the syntactic level, the possibilistic base should contain the formula $(s\vee d\vee o)$ with the highest weight. Then, immediate preferred interpretations are those which only falsify one literal. This means that B_{K_3} contains the three formulas $(s\vee o),(s\vee d)$, $(o\vee d)$ with a lower weight (which correspond to the three possible ways of removing one literal from the disjunction $s\vee d\vee o$). Next, the more preferred interpretations are those which falsify two literals. This leads to add the literals o,s,d with a smaller weight in the possibilistic base. To summarize, B_{K_3} should be of the form:
$B_{K_3}=\{(s\vee d\vee o,\alpha);(s\vee o,\beta);(s\vee d,\beta);(o\vee d,\beta);(s,\delta);(o,\delta);(d,\delta)\}$, with $\alpha>\beta>\delta$.

We can easily check that:
$\forall\omega, \pi_{B_{K_3}}(\omega)>\pi_{B_{K_3}}(\omega')$ iff $d(\omega,K_3)<d(\omega',K_3)$.

Moreover, if we let $\alpha=1-\varepsilon^3$, $\beta=1-\varepsilon^2$, $\delta=1-\varepsilon$, where ε is a very small number, then

- B_{K_3} is the result of combining the three elementary bases $\Sigma_s=\{(s,1-\varepsilon)\}$, $\Sigma_o=\{(o,1-\varepsilon)\}$, $\Sigma_d=\{(d,1-\varepsilon)\}$ with the product operator.
- $\forall\omega\in\Omega, \pi_{B_{K_3}}(\omega)=\varepsilon^{d(\omega,K_3)}$.

These remarks can be generalized. First, we need the following:

Lemma 2 *Let $K=\{l_i : i=1,n\}$ be a set of literals. Let $\Sigma_i=\{(l_i,1-\varepsilon)\}$ be n possibilistic bases with one formula. Then combining Σ_i's with the product operator leads to the following base:*

$$B_K=\{(D_j(K),1-\varepsilon^j) : j=1,n\},$$

where $D_j(K)$ are all possible disjunctions of size j from K.

Example 1 (continued)
Let us consider the base $K_3=\{s,d,o\}$.

We have the combination of Σ_s and Σ_d with the product leads to:
$\{(s,1-\varepsilon)\}\cup\{(d,1-\varepsilon)\}\cup\{(s\vee d,1-\varepsilon^2)\}$,
combining again this resulting base with Σ_o leads to:
$\{(s,1-\varepsilon)\}\cup\{(d,1-\varepsilon)\}\cup\{(o,1-\varepsilon)\}\cup\{(s\vee d,1-\varepsilon^2);(s\vee o,1-\varepsilon^2);(d\vee o,1-\varepsilon^2)\}\cup\{(s\vee d\vee o,1-\varepsilon^3)\}$.

The following proposition explicits the encoding of Dalal's distance in possibilistic logic.

Proposition 10 *Let K be a set of literals. Let B_K be the base obtained using Lemma 2. Then,*

$$\pi_{B_K}(\omega)=\varepsilon^{d(\omega,K)}.$$

Now let consider the case where K is a *general* knowledge base which is put under a disjunctive normal form i.e., $K=C_1\vee...\vee C_n$. Note first that (Lin 1996):

$$d(\omega,K)=min_{i=1,n}d(\omega,C_i) \qquad (2)$$

Moreover, from Proposition 10, we have shown that it is possible to associate to each C_i a possibilistic base B_{C_i} where $\forall\omega, \pi_{B_{C_i}}(\omega)=\varepsilon^{d(\omega,C_i)}$ (3)

Then from (2) and (3) it can be easily guessed that:
$\pi_K(\omega)=max_{i=1,n}\pi_{B_{C_i}}(\omega)=max_{i=1,n}\varepsilon^{d(\omega,C_i)}=\varepsilon^{min_{i=1,n}d(\omega,C_i)}=\varepsilon^{d(\omega,K)}$.

Therefore the possibilistic base B_K associated to K is the result of combining the possibilistic bases B_{C_i}'s associated to each conjunct C_i of K with the maximum operator.

Proposition 11 *Let $K=\{C_1\vee\cdots\vee C_n\}$ be a classical base, and B_{C_i}'s be the possibilistic bases associated to C_i's obtained from Lemma 2. Let B_K be the combination of B_{C_i}'s with the maximum operator. Then,*

$$\pi_{B_K}(\omega)=\varepsilon^{d(\omega,K)}.$$

Now, due to the particular structure of B_{C_i}'s, the following proposition gives an equivalent rewriting of B_K, without computing explicitly B_{C_i}'s:

Proposition 12 *Let $K=\{C_1\vee...\vee C_n\}$.*
Let $m=min_{i=1,n}(|C_i|)$. Then, B_K obtained from Prposition 11 is equivalent [1] to the following possibilistic base:

$$\{(D_j,1-\varepsilon^j) : j=1,m\},$$

where D_j is a clause (i.e., disjunction of literals) containing exactly j literals from each C_i of K.

The number m in the previous proposition denotes the number of layers in the possibilistic base associated to K.

Example 1 (continued)
Let us put the bases K_1,K_2 and K_3 in a disjunctive form: $K_1=\{C_1,C_2\}$; $K_2=\{C_3,C_4\}$ and $K_3=\{C_5\}$ such that $C_1=\{s,\neg d\}$; $C_2=\{o,\neg d\}$; $C_3=\{\neg s,d,\neg o\}$; $C_4=\{\neg s,\neg d,o\}$ and $C_5=\{s,d,o\}$.
For K_1, we have $|C_1|=|C_2|=2$. Then, $m=2$. From

[1] The equivalence is in the sense of Definition 3.

Proposition 12:

$$B_{K_1} = \{(s \vee \neg d \vee o, 1-\varepsilon^2); (s \vee \neg d, 1-\varepsilon);$$
$$(s \vee o, 1-\varepsilon); (\neg d \vee o, 1-\varepsilon); (\neg d, 1-\varepsilon)\}$$
$$\equiv_s \{(s \vee \neg d \vee o, 1-\varepsilon^2); (s \vee o, 1-\varepsilon); (\neg d, 1-\varepsilon)\}.$$

The formulas $(s \vee \neg d, 1-\varepsilon)$ and $(\neg d \vee o, 1-\varepsilon)$ are removed from the possibilistic base since they are subsumed by $(\neg d, 1-\varepsilon)$.
With a similar computation we get:

$$B_{K_2} = \{(\neg s \vee d \vee o, 1-\varepsilon^2); (\neg s \vee \neg d \vee \neg o, 1-\varepsilon^2);$$
$$(\neg s, 1-\varepsilon); (d \vee o, 1-\varepsilon); (\neg d \vee \neg o, 1-\varepsilon)\}.$$

B_{K_3} has been previously computed.

Encoding the global distance in the possibilistic logic

Semantic encoding of the global distance

In this subsection, given an information set $E = \{K_1, \cdots, K_n\}$, a classical merging operator Δ, and $\mathcal{B} = \{B_{K_1}, \cdots, B_{K_n}\}$ the possibilistic counterpart of E obtained from Proposition 12, we show that it is possible to define a possibilistic merging operator $\oplus$ applied on $\mathcal{B}$ and yielding to $\mathcal{B}_\oplus$ such that $\pi_{\mathcal{B}_\oplus}$, the possibility distribution associated to $\mathcal{B}_\oplus$, is function of d_Δ. Namely,

$$\pi_{\mathcal{B}_\oplus}(\omega) = f(d_\Delta(\omega, E)).$$

Let us illustrate this encoding for the majority operator Σ at the semantic level.

Proposition 13 *Let $E = \{K_1, \cdots, K_n\}$ be an information set. Let $\mathcal{B} = \{B_{K_1}, \cdots, B_{K_n}\}$ be the possibilistic counterpart of E. Let $\Delta_\Sigma(E)$ be the result of combining K_i's with Σ operator. Let $\pi_{B_{K_i}}$ be the possibility distribution associated to B_{K_i}.*
Let $\pi_{\mathcal{B}Pro}$ be the result of combining $\pi_{B_{K_i}}$'s with the product operator Pro. Then,

$$\pi_{\mathcal{B}Pro}(\omega) = \varepsilon^{d_\Sigma(\omega,E)}.$$

The proof can be easily checked.
Recall that $d_\Sigma(\omega, E) = \Sigma_{i=1,n} d(\omega, K_i)$.
Then, $\pi_{\mathcal{B}Pro}(\omega) = \pi_{B_{K_1}} * \cdots * \pi_{B_{K_n}} = \varepsilon^{d(\omega,K_1)} * \cdots * \varepsilon^{d(\omega,K_n)} = \varepsilon^{\Sigma_{i=1,n} d(\omega,K_i)} = \varepsilon^{d_\Sigma(\omega,E)}$.

With a similar way we get the possibilistic counterparts of *max* and *ws*.

The following table gives which possibilistic operator $\oplus$ should be used for recovering classical merging operator Δ:

Δ	$\oplus$
Σ	Pro
ws	weighted Pro
max	min
$gmax$	Pro (with changing scale)

Table 3

where *weighted Pro* is simply the product applied repeatedly. Since the k_i's are integers, each base is repeated n times if $k_i = n$.

The counterpart of $gmax$ needs a transformation of the scale of the weights given to the literals.

Syntactic counterpart of the semantic encoding

We have above shown that given an information set $E = \{K_1, \cdots, K_n\}$ and a classical operator Δ, we can define $\mathcal{B} = \{B_{K_1}, \cdots, B_{K_n}\}$ such that combining $\pi_{B_{K_i}}$'s (the possibility distributions associated to B_{K_i}'s) with some $\oplus$, the counterpart of Δ according to Table 2, leads to the following relation:

$$\pi_{\mathcal{B}_\oplus}(\omega) = \varepsilon^{d_\Delta(\omega,E)} \qquad (4).$$

Our aim is to give a direct syntactic computation of $\Delta(E)$ without computing $\pi_{\mathcal{B}_\oplus}$.
Recall that the models of $\Delta(E)$ are the interpretations which minimize $d_\Delta(\omega, E)$. Then, from (4), these interpretations should maximize $\pi_{\mathcal{B}_\oplus}$ since $\varepsilon < 1$.
Besides, given a possibility distributions π_B associated to a possibilistic base B, it can be checked that the interpretations which maximize π_B are the models of $\rho(B)$, the consistent part of B.
Now, $\pi_{\mathcal{B}_\oplus}$ is the result of combining $\pi_{B_{K_i}}$'s with $\oplus$. Then, the possibilistic base, denoted by $\mathcal{B}_\oplus$, associated with $\pi_{\mathcal{B}_\oplus}$ is simply the result of combining B_{K_i}'s with $\oplus$. Hence, $\Delta(E)$ is simply $\rho(\mathcal{B}_\oplus)$.

Proposition 14 *Let $E = \{K_1, \cdots, K_n\}$ be an information set, and $\Delta(E)$ be the result of combining K_i's with Δ.*
Let $\mathcal{B} = \{B_{K_1}, \cdots, B_{K_n}\}$ be the possibilistic counterpart of E obtained from Proposition 12.
Let $\oplus$ be the syntactic counterpart of Δ obtained from Table 2, and $\mathcal{B}_\oplus$ be the result of combining B_{K_i}'s with $\oplus$. Then,

$$\Delta(E) \equiv \rho(\mathcal{B}_\oplus).$$

Example 1 (continued)
Let us consider the majority operator Σ. Let B_{K_1},B_{K_2} and B_{K_3} be the possibilistic counterpart of K_1, K_2 and K_3 computed in last example. Then to compute the possibilistic base associated to the combination of $B_{K_1}, B_{K_2}, B_{K_3}$ with majority operator, we combine B_{K_i}'s with the product operator. Since the product operator is associative, let us first compute $\mathcal{B}'$ the base resulting from the combining B_{K_1} and B_{K_2}.

$$\mathcal{B}' = \{(\neg s \vee \neg d \vee \neg o, 1-\varepsilon^3); (s \vee d \vee o, 1-\varepsilon^2); (\neg s \vee \neg d, 1-\varepsilon^2);$$
$$(\neg d \vee \neg o, 1-\varepsilon^2); (s \vee \neg d \vee o, 1-\varepsilon^2); (\neg s \vee d \vee o, 1-\varepsilon^2);$$
$$(s \vee o, 1-\varepsilon); (\neg d, 1-\varepsilon); (\neg s, 1-\varepsilon); (d \vee o, 1-\varepsilon)\}.$$

Then combining $\mathcal{B}'$ and B_{K_3} leads to after removing subsumed formulas:

$$\mathcal{B}_\oplus = \{(s \vee d \vee o, 1-\varepsilon^5); (s \vee \neg d \vee o, 1-\varepsilon^4); (\neg s \vee d \vee o, 1-\varepsilon^4);$$
$$(\neg s \vee \neg d \vee o, 1-\varepsilon^3); (s \vee \neg d \vee \neg o, 1-\varepsilon^3);$$
$$(\neg s \vee \neg d \vee \neg o, 1-\varepsilon^3); (\perp, 1-\varepsilon^2)\}.$$

Note that we obtain a contradiction here with a weight $1-\varepsilon^2$ which expresses conflict between sources.
Note that $Inc(\mathcal{B}_\oplus)$ is always of the from $1-\varepsilon^j$ where j is the minimal weight in the global distance obtained from merging E by Δ. For instance, we can check that in this example we have $Inc(\mathcal{B}_\oplus) = 1-\varepsilon^2$ and indeed 2 is the minimal distance in Table 1 (for Σ). When

$j = 0$ namely $Inc(\mathcal{B}_{\oplus})= 0$, this simply means that $\wedge E = K_1 \wedge ... \wedge K_n$ is consistent.
Now, to recover $\Delta_{\Sigma}(\{K_1, K_2, K_3\})$ we simply compute $\rho(\mathcal{B}_{\oplus})$, namely $\rho(\mathcal{B}_{\oplus})= \{s \vee d \vee o, s \vee \neg d \vee o, \neg s \vee d \vee o, \neg s \vee \neg d \vee o, s \vee \neg d \vee \neg o, \neg s \vee \neg d \vee \neg o\} \equiv \{\neg d, o\}$. Recall that $\rho(\mathcal{B}_{\oplus})$ contains classical formulas having a weight greater than the inconsistency degree of $\mathcal{B}_{\oplus}$ (here equal to $1 - \varepsilon^2$).
Clearly we can check that $[\rho(\mathcal{B}_{\oplus})]= \{\omega_1, \omega_5\}$ which has been obtained in above examples.

Remark 1 *Note that the advantage of giving a possibilistic base instead of the classical base $\rho(\mathcal{B}_{\oplus})$ as a result of the merging is the capability of iteration for associative operators, which is not possible with classical merging operators. Indeed, in the above example we have $(B_{K_1} \oplus B_{K_2}) \oplus B_{K_3} = B_{K_1} \oplus B_{K_2} \oplus B_{K_3}$, and this is general.*

Adding integrity constraints

We consider now the merging of classical knowledge bases with integrity constraints. The integrity constraints are requirements that the merged base must satisfy. Let $E = \{K_1, ..., K_n\}$ be the information set to merge, and μ be a set of constraints. We denote by $\Delta(\mu, E)$ the result of merging with respect to an operator Δ. The operation of fusion follows the same steps as Δ. The only change appears in selecting models of $\Delta(\mu, E)$ which focuses on models of μ which are minimal in $\leq_E^{\Delta}$, namely

$$[\Delta(E)] = min([\mu], \leq_E^{\Delta}).$$

It means that we only consider interpretations which satisfy the set of integrity constraints. Since the formulas of μ must be satisfied, they are considered as fully reliable in the possibilistic framework. The possibilistic propositional base associated to a set of integrity constraints is then: $B_{\mu} = \{(\phi_i, 1) : \phi_i \in \mu\}$.

Proposition 15 *Let E and μ be respectively an information set and a set of integrity constraints, and Δ be a classical merging operator. Let $\mathcal{B}_{\oplus}$ be the possibilistic counterpart of E. Then the possibilistic base associated to E and μ is simply $\mathcal{B}_{\oplus} \cup B_{\mu}$.*

Conclusion

The paper has shown how classical fusion operators can be embedded in the framework of possibilistic logic where formulas are associated with priority levels. This has several benefits: *i-* it allows for the iteration of the fusion process in a satisfactory way; *ii-* the explicit expression of priorities, as suggested by the example, may be useful and meaningful in practice; *iii-* a syntactic counterpart of the fusion operators can be systematically defined. Besides, the study of fusion operators in the possibilistic framework from the point of view of normative postulates can be found in (Benferhat et al. 2000).

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