Linköping Electronic Articles in
Computer and Information Science
Vol. 7(2002): nr 7

Disjunctive Explanations in Abductive Logic Programming

Katsumi Inoue
Department of Electrical and Electronics Engineering
Kobe University
Rokkodai, Nada, Kobe 657-8501, Japan
inoue@eedept.kobe-u.ac.jp

Chiaki Sakama
Department of Computer and Communication Sciences
Wakayama University
Sakaedani, Wakayama 640-8510, Japan
sakama@sys.wakayama-u.ac.jp

Linköping University Electronic Press
Linköping, Sweden

http://www.ep.liu.se/ea/cis/2002/007/

Published on October 15, 2002 by
Linköping University Electronic Press
581 83 Linköping, Sweden

Linköping Electronic Articles in
Computer and Information Science
ISSN 1401-9841
Series editor: Erik Sandewall

© *2002 Katsumi Inoue and Chiaki Sakama*
Typeset by the author using LaTeX
Formatted using étendu *style*

Recommended citation:
<Author>. <Title>. Linköping Electronic Articles in Computer and Information Science, Vol. 7(2002): nr 7.
http://www.ep.liu.se/ea/cis/2002/007/. *October 15, 2002.*

This URL will also contain a link to the author's home page.

The publishers will keep this article on-line on the Internet (or its possible replacement network in the future) for a period of 25 years from the date of publication, barring exceptional circumstances as described separately.

The on-line availability of the article implies a permanent permission for anyone to read the article on-line, to print out single copies of it, and to use it unchanged for any non-commercial research and educational purpose, including making copies for classroom use. This permission can not be revoked by subsequent transfers of copyright. All other uses of the article are conditional on the consent of the copyright owner.

The publication of the article on the date stated above included also the production of a limited number of copies on paper, which were archived in Swedish university libraries like all other written works published in Sweden. The publisher has taken technical and administrative measures to assure that the on-line version of the article will be permanently accessible using the URL stated above, unchanged, and permanently equal to the archived printed copies at least until the expiration of the publication period.

For additional information about the Linköping University Electronic Press and its procedures for publication and for assurance of document integrity, please refer to its WWW home page: http://www.ep.liu.se/
or by conventional mail to the address stated above.

Abstract

Given some evidence or observation, abduction infers candidate hypotheses to explain such an observation. In this paper, we consider the case in which multiple explanations exist but we cannot select one from them. Our approach leaves the decision indefinite, but offers a consistent way to theory changes. A *disjunctive explanation*, which is a disjunction of possible alternative explanations, is introduced for this purpose. For abduction, such a disjunction is useful when obtained explanations are incorporated or assimilated into the current knowledge base. The assimilated knowledge base preserves the intended semantics from the collection of all possible updated knowledge bases. The merit of disjunctive explanations is that we only need one current knowledge base at a time, still keeping every possible change in a single state. This method is extended to accomplish an update when there are multiple ways to remove old unnecessary hypotheses from the knowledge base. The proposed framework is also well applied to view updates in disjunctive databases.

1 Introduction

The task of abduction is to infer explanations accounting for an observation. *Abductive logic programming* (ALP) [23] has been widely used to declaratively specify a variety of problems in AI including updates in data and knowledge bases, belief revision, diagnosis, causal theory, and default reasoning. One of the most significant issues in abductive logic programming is to develop a reasonable method for *knowledge assimilation*, which incorporates obtained explanations into the current knowledge base.

In general, we may encounter multiple explanations for the given observation. When there are multiple explanations of an observation G, we see that the disjunction of these explanations also accounts for G. In this paper, we formalize this idea by extending the notion of "explanation" to more general one than the traditional framework of ALP. Then, this paper offers a solution to the problem of knowledge assimilation by considering *disjunctive explanations* whenever multiple explanations exist.

Suppose that we are given the background knowledge K, a set of abducibles $\mathcal{A}$ and an observation G. Then, each set E of instances of elements from $\mathcal{A}$ satisfying that (i) $K \cup E \models G$ and (ii) $K \cup E$ is consistent, is called an *elementary explanation* in this paper. Then, any disjunction of elementary explanations is called an *explanation*. The reason why we use the term "explanation" for a disjunction of (elementary) explanations is that, if $\{e_1\}$ and $\{e_2\}$ are (elementary) explanations of G, then in first-order logic or logic programming with the answer set semantics, $e = e_1 \vee e_2$ satisfies that (i) $K \cup \{e\} \models G$ and (ii) $K \cup \{e\}$ is consistent.

The use of disjunctive explanations is quite natural when the background knowledge K is represented in a *disjunctive logic program*. Also, disjunctive explanations are useful in various applications involving abduction. In the following, we list typical applications of disjunctive explanations.

Weakest explanations. In abduction, we usually seek for *least presumptive* or *weakest explanations*. Such an explanation is often called a *weakest sufficient condition* [29]. When $\{e_1\}$ and $\{e_2\}$ are minimal elementary explanations of G, where the minimality is defined in terms of the set inclusion relation, each explanation $\{e_i\}$ $(i = 1, 2)$ is most preferred in traditional formalizations of abduction because $\{e_i\}$ is weaker than any non-minimal explanation like $\{e_1, e_2\}$, i.e., $\{e_1, e_2\} \models e_i$. However, the disjunctive explanation $\{e_1 \vee e_2\}$ is much weaker, i.e., $\{e_i\} \models e_1 \vee e_2$. For another example, when $\{a, b\}$ and $\{c\}$ are the two minimal elementary explanations, $\{a \vee c, b \vee c\}$ is the weakest explanation because we see that $(a \wedge b) \vee c \equiv (a \vee c) \wedge (b \vee c)$.

Skeptical reasoning and minimization. Abduction can be used to implement skeptical reasoning in abductive and nonmonotonic reasoning [17]. In query answering from *circumscription* [13], we often need disjunctive explanations. For example, if both $\neg ab1$ and $\neg ab2$ can explain g and the clause $ab1 \vee ab2$ can be entailed from the background theory, then the disjunction $\neg ab1 \vee \neg ab2$ skeptically explains g. This is verified by failing to explain the negation $ab1 \wedge ab2$ of the disjunctive explanation $\neg ab1 \vee \neg ab2$. A minimization principle with disjunctive explanations is also employed in abduction from *causal theories* [26].

Inductive Logic Programming (ILP). Construction of inductive hypotheses in ILP requires taking disjunction of hypotheses [16]. For example, let $\{human(socrates)\}$ be a background theory, and $mortal(socrates)$ an observation. Also, let $\{human(x), \neg human(x), mortal(x), \neg mortal(x)\}$ be the abducibles, that is, any instance of $human(x)$ or $mortal(x)$ can be hypothesized as either a positive or negative literal. Then, in most formalizations of ALP, explanations are given as $\{mortal(socrates)\}$ or $\{mortal(x)\}$, where the latter is a non-minimal explanation. However, we prefer $\neg human(x) \vee mortal(x)$ as a better hypothesis in ILP. This hypothesis actually satisfies the principle of *inverse entailment* [32].

In another case, suppose that we learned two rules for the predicate p as:

$$\begin{aligned} C1(x) &\supset p(x), \\ C2(x) &\supset p(x). \end{aligned}$$

Then, taking the disjunction of these two hypotheses corresponds to computing the *greatest specialization under implication* [33]:

$$C1(x) \wedge C2(y) \supset p(x) \vee p(y).$$

This technique may be used as a specialization technique in ILP.

Negative (anti-)explanation. In *extended abduction* [19], we may want to remove abducible facts from the background theory. For example, suppose that the program is given as:

$$\begin{aligned} g &\leftarrow not\, p, \\ p &\leftarrow a, \\ p &\leftarrow b, \\ a; b &\ , \end{aligned}$$

and the abducibles are $\{a, b\}$. Then, to explain g (or *insert* g or *delete* p in *view update*), it is necessary to remove the disjunction $a; b$ from the program. However, the previous framework of extended abduction [19, 15] cannot do that, because only instances of elements from the abducibles can be manipulated. Here, removing $\{a\}$ or $\{b\}$ or $\{a, b\}$ cannot be successful because neither a nor b is in the program.

Knowledge base update. Adapting alternative solutions for an update request to the background theory usually results in multiple alternative new states. The disjunction of these solutions offers a good solution representing every possible change in a single state [7, 8, 34]. This technique reduces the size of knowledge bases through a sequence of updates and keeps only one current knowledge base at a time. This application—knowledge base update—is particularly important when we want to assimilate explanations into our current knowledge base. While knowledge assimilation is one of the most significant problems in ALP [25, 23], not much work has been reported so far.

In the rest of this paper, we explore the theory of disjunctive explanations in the context of abductive logic programming. We will prove that the assimilated program with disjunctive explanations preserves all and only minimal answer sets from the collection of all possible updated programs.

We also introduce disjunctive explanations into the framework of *extended abduction* [19], where both addition and removal of hypotheses are allowed to explain or unexplain an observation. When there are multiple preferred explanations involving removal of hypotheses, assimilating them into one knowledge base is much more difficult than in the case of normal abduction which only adds hypotheses. For example, to explain p (or to insert p in view update) from

$$\begin{array}{lll} p & \leftarrow & not\, a, \\ p & \leftarrow & not\, b, \\ a & , & \\ b & , & \end{array}$$

with the abducibles $\{a, b\}$, removing $\{a\}$, $\{b\}$, or $\{a, b\}$ is too strong. In fact, we may remove either $\{a\}$ or $\{b\}$ without committing which one is actually removed. However, this indefiniteness cannot be represented in any known current technique.

It is known that extended abduction can be used to formalize various update problems in AI and databases [19, 21, 35]. That is, an *insertion/deletion* of a fact G into/from a database is accomplished by a minimal *explanation/anti-explanation* of G. Then, the notion of disjunctive explanations in this paper can also be applied to update problems in databases. In particular, the *view update* problem in *disjunctive databases*, i.e., deductive databases possibly containing disjunctions which represent indefinite or uncertain information, can also be realized within the proposed framework. When we build a database in real-life situations, a database is likely to include such disjunctive facts. Developing an update technique in disjunctive databases is therefore important from practical viewpoints. However, disjunctive databases are more expressive than Datalog [6], and view updates in disjunctive databases are more difficult than the case of Datalog. In fact, there are few studies on the subject of updating disjunctive databases and many problems have been left open. Hence, with our proposed framework, we can make advances in studies of view updates in disjunctive databases.

This paper is an extended version of the paper [22]. The rest of this paper is organized as follows. Section 2 reviews a framework of disjunctive logic programming and its answer set semantics. Section 3 introduces the abductive framework considering disjunctive explanations. Section 4 extends the notion of disjunctive explanations to extended abduction which allows removal of abducibles from a program. Section 5 discusses related issues, and Section 6 is a conclusion. This paper contains all complete proofs which are omitted in [22].

2 Disjunctive Programs

A knowledge base or database is represented in a *(disjunctive) program*, which consists of a finite number of *rules* of the form:

$$F_1; \cdots; F_l \leftarrow L_1, \ldots, L_m, \; not\, L_{m+1}, \ldots, not\, L_n \qquad (1)$$

where each L_i is a literal ($n \geq m \geq 0$), *not* is *negation as failure* (NAF), and F_i ($l \geq 0$) is a conjunction of literals:

$$F_i = L_{i,1}, \ldots, L_{i,k_i} \quad (1 \leq i \leq l, \; k_i \geq 1).$$

The symbols ; and , respectively represent a disjunction and a conjunction, and are often written also as $\vee$ and $\wedge$, respectively. A rule with variables stands for the set of its ground instances. We assume that function symbols never appear in a program, which implies that a number of the ground instances of a variable is finite.[1] The left-hand side of the rule is the *head*, and the right-hand side is the *body*. A rule with the empty head is an *integrity constraint*. Any rule with the empty body $F_1; \cdots; F_l \leftarrow$ is called a *fact* and is also written as $F_1; \cdots; F_l$ without the symbol $\leftarrow$. A fact is *disjunctive* if $l > 1$. In this paper, we often identify a conjunction of literals $L_1, \ldots, L_k$ with the set of its conjuncts $\{L_1, \ldots, L_k\}$. Also, any disjunction like $L; L$ is identified with its merged form L in this paper.

A rule of the form (1) is a generalization of a rule in an *extended disjunctive program* (EDP) defined by [11]. The generalized part appear in the head: a *disjunctive normal form* (DNF) formula $F_1; \cdots; F_l$ is allowed in (1),[2] while the head of an original EDP rule is just a disjunction of literals $L_1; \cdots; L_l$, which represents the case of $k_i = 1$ for every $i = 1, \ldots, l$ in (1). Such rules with DNF heads were introduced for EDPs without NAF in [18]. A rule of the form (1) can be equivalently converted into $k_1 \times \cdots \times k_l$ rules with ordinary heads:

$$L_{1,j_1}; \cdots; L_{l,j_l} \leftarrow L_1, \ldots, L_m, \mathit{not}\, L_{m+1}, \ldots, \mathit{not}\, L_n, \tag{2}$$

for every $j_1 = 1, \ldots, k_1, \ \ldots, \ j_l = 1, \ldots, k_l$ [20]. Thus, we also call a program consisting of rules of the form (1) an *EDP*, and often identify a rule of the form (1) with the collection of its converted form (2) in a program.

Any program K is divided into two parts,

$$K = \mathcal{I}(K) \cup \mathcal{F}(K), \text{ where } \mathcal{I}(K) \cap \mathcal{F}(K) = \emptyset$$

$\mathcal{F}(K)$ denotes the set of facts in K, and $\mathcal{I}(K) = K \setminus \mathcal{F}(K)$. When we consider a database written as a program, $\mathcal{I}(K)$ represents an *intensional database*, and $\mathcal{F}(K)$ is an *extensional database*.

The semantics of a program is given by its *answer sets*. The following definition is due to [11, 20], and the equivalent definition can also be obtained from [28]. First, let K be an EDP without NAF (i.e., $m = n$) and $S \subseteq \mathcal{L}$, where $\mathcal{L}$ is the set of all ground literals in the language of K. Then, S is an *answer set* of K if S is a minimal set satisfying the conditions:

1. For each ground rule

$$F_1; \cdots; F_l \leftarrow L_1, \ldots, L_m$$

from K, if $\{L_1, \ldots, L_m\} \subseteq S$ then $F_i \subseteq S$ for some F_i $(1 \leq i \leq l)$;

2. If S contains a pair of complementary literals L and $\neg L$, then $S = \mathcal{L}$.

Second, given *any* EDP K (with NAF) and $S \subseteq \mathcal{L}$, consider the EDP without NAF K^S obtained as follows: a rule $F_1; \cdots; F_l \leftarrow L_1, \ldots, L_m$ is in K^S iff there is a ground rule of the form (1) from K such that $\{L_{m+1}, \ldots, L_n\} \cap S = \emptyset$. Then, S is an *answer set* of K if S is an answer set of K^S.

An answer set is *consistent* if it is not $\mathcal{L}$. A program is *consistent* if it has a consistent answer set. Note that every answer set S of any EDP is

[1] This assumption is necessary only for later use in representing explanation closures of an observation in first-order logic (Definition 3.4).

[2] In this paper, rules of the form (1) are mostly used to represent disjunctive explanations, which are in the form of (1) without bodies.

minimal [11], that is, no other answer set S' of K satisfies that $S' \subset S$. The set of all answer sets of K is written as $\mathcal{AS}(K)$. For a literal L, K *entails* L, written $K \models L$, if $L \in S$ for every $S \in \mathcal{AS}(K)$.

Note that we could consider more general form of programs allowing *nested expressions* [28]. In that case, an answer set of a program is possibly non-minimal, and some properties in the rest of this paper has to be changed slightly. An EDP is called an *extended logic program* (ELP) if it contains no disjunction, i.e., $l \leq 1$ for any rule of the form (1), and an ELP is called a *normal logic program* (NLP) if every L_i is an atom.

We provide an entailment relationship between programs as follows. Let R and R' be sets of rules of the form (1). We write $R \models R'$ if for any $S \in \mathcal{AS}(R)$, there exists $S' \in \mathcal{AS}(R')$ such that $S' \subseteq S$. In this case, we say that R' is *weaker than* R. For example, $\{a, b\} \models \{a\} \models \{a; b\}$. We also say that R and R' are *equivalent* if $\mathcal{AS}(R) = \mathcal{AS}(R')$.[3]

We also define an alternative ordering between rule sets. Let R and R' be rule sets, and K a program. We say that R' is *less presumptive (relative to K) than* R if $K \cup R \models K \cup R'$. We also say that R and R' are *equivalent relative to* K if $\mathcal{AS}(K \cup R) = \mathcal{AS}(K \cup R')$.

3 Disjunctions in Normal Abduction

An *abductive program* is a pair $\langle K, \mathcal{A} \rangle$, where both K and $\mathcal{A}$ are EDPs. Each element of $\mathcal{A}$ and its any instance is called an *abducible*. When a rule is an abducible, it is called an *abducible rule*. Such an abducible rule can be associated with a unique literal called the *name* [14]. Then, with this naming technique, we can always assume in this paper that the abducibles $\mathcal{A}$ of an abductive program $\langle K, \mathcal{A} \rangle$ is a set of literals. Moreover, we assume without loss of generality that, any rule from K having an abducible in its head is always a fact consisting of abducibles only.[4] In abduction, we are given an *observation* G to be explained or unexplained. Without loss of generality, such an observation is assumed to be a non-abducible ground literal [20].

We firstly consider *normal abduction*, and later in Section 4 extend our framework by considering *extended abduction* [19].

Definition 3.1 Let $\langle K, \mathcal{A} \rangle$ be an abductive program and G an observation. A set E is an *elementary explanation of G (wrt $\langle K, \mathcal{A} \rangle$)* if

1. E is a set of ground instances of elements from $\mathcal{A}$,
2. $K \cup E \models G$, and
3. $K \cup E$ is consistent.

Note here that we use the term "elementary explanation" instead of just calling "explanation". The latter term is reserved for the next definition.

Definition 3.2 A DNF formula $e_1; \cdots; e_n$ $(n > 0)$ is a *(disjunctive) explanation* of G if each e_i $(i = 1, \ldots, n)$ is the conjunction of the elements in an elementary explanation of G.

[3] This equivalence is called *weak equivalence* in [27].

[4] A similar assumption is usually used in literature, e.g., [23]. If there is a fact containing both an abducible a and a non-abducible or there is a rule containing an abducible a in its head and a non-empty body, then such an abducible a is made a non-abducible by introducing a rule $a \leftarrow a'$ with a new abducible a' and then replacing a with a' in every fact consisting abducibles only.

Any elementary explanation E is a disjunctive explanation since we identify the conjunction of the literals in E with E. Similarly, when $E_1, \ldots, E_n$ are elementary explanations of G, we say $E_1 \vee \cdots \vee E_n$ is a (disjunctive) explanation of G. When a program K is augmented with a single disjunctive explanation E, we will often write $K \cup E$ instead of $K \cup \{E\}$. Such disjunctive explanations deserve to be called "explanations" as the next proposition holds.

Proposition 3.1 *Let E be a (disjunctive) explanation of G wrt $\langle K, \mathcal{A} \rangle$. Then, $K \cup E \models G$ and $K \cup E$ is consistent.*

Proof: Let e_1 and e_2 be elementary explanations of G wrt $\langle K, \mathcal{A} \rangle$. We prove that the disjunctive explanation $e_1 \vee e_2$ satisfies that $K \cup \{e_1 \vee e_2\} \models G$ and $K \cup \{e_1 \vee e_2\}$ is consistent. This proof can be applied to a disjunctive explanation constructed with more than two elementary explanations. Without loss of generality, we also assume that $e_1 \not\models e_2$ and $e_2 \not\models e_1$.

Considering the program $K' = K \cup \{e_1 \vee e_2\}$, K' is consistent if $K \cup e_1$ or $K \cup e_2$ is consistent. Since e_1 and e_2 are elementary explanations of G, both $K \cup e_1$ and $K \cup e_2$ are consistent. Hence, $K \cup \{e_1 \vee e_2\}$ is consistent. Let S be any consistent answer set of $K \cup \{e_1 \vee e_2\}$. Then, by the definition of answer sets, either $e_1 \subseteq S$ or $e_2 \subseteq S$ holds. Again, since e_1 and e_2 are elementary explanations of G, both $K \cup e_1 \models G$ and $K \cup e_2 \models G$ hold. Then, G is also true in S. Hence, $K \cup \{e_1 \vee e_2\} \models G$. □

Note that a disjunctive explanation is in the form of a disjunction of conjunctions of abducibles, that is, a DNF formula. There exists a logically equivalent explanation in the form of a CNF formula, i.e., a conjunction of disjunctions of abducibles. To specify such CNF explanations, we consider an alternative formalization of abduction with the enlarged hypothesis space which consists of *disjunctive hypotheses*. Given an abductive program $\langle K, \mathcal{A} \rangle$, the *enlarged abducible set*, written $\mathcal{D}(\mathcal{A})$, consists of every disjunction of abducibles from $\mathcal{A}$.[5] Then, we can define an abductive program $\langle K, \mathcal{D}(\mathcal{A}) \rangle$, in which we can abduce any disjunction of abducibles to explain an observation. Also, we can define elementary explanations for the abductive program $\langle K, \mathcal{D}(\mathcal{A}) \rangle$. However, we do not need the notion of disjunctive explanations for $\langle K, \mathcal{D}(\mathcal{A}) \rangle$ since such disjunctions can be represented as elementary explanations from $\mathcal{D}(\mathcal{A})$.

The next proposition shows such a fundamental property, which represents the relationship between the disjunctive explanations of G wrt $\langle K, \mathcal{A} \rangle$ and the elementary explanations of G wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$.

Proposition 3.2 *Let $\langle K, \mathcal{A} \rangle$ be an abductive program and G an observation. If D is an elementary explanation of G wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$, then there is a disjunctive explanation E of G wrt $\langle K, \mathcal{A} \rangle$ such that D and E are equivalent relative to K. Conversely, if E is a disjunctive explanation of G wrt $\langle K, \mathcal{A} \rangle$, then there is an elementary explanation D of G wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$ such that E and D are equivalent relative to K.*

Proof: Suppose that $D = \{d_1, \ldots, d_n\}$ is an elementary explanation of G wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$, where $d_i \in \mathcal{D}(\mathcal{A})$ for $i = 1, \ldots, n$. Let $f_1; \cdots; f_m$ be a DNF formula that is logically equivalent to the CNF formula D. Further, let $e_1; \cdots; e_k$ be a sub-disjunction of $f_1; \cdots; f_m$ $(k \leq m)$ such that

$$\{e_1, \ldots, e_k\} = \{f_i \mid K \cup f_i \text{ is consistent } (1 \leq i \leq m)\} \subseteq \{f_1, \ldots, f_m\}.$$

[5] Again, any disjunction like $a; a; b$ is identified with its merged form $a; b$ in $\mathcal{D}(\mathcal{A})$.

Since $K \cup \{f_1; \cdots; f_m\} \models G$ holds by assumption, $K \cup \{e_1; \cdots; e_k\} \models G$ holds. By the construction of e_j $(1 \leq j \leq k)$, $K \cup e_j$ is consistent and hence $K \cup e_j \models G$. Thus, each e_j is an elementary explanation of G wrt $\langle K, \mathcal{A} \rangle$. Therefore, $e_1; \cdots; e_k$ is a disjunctive explanation of G.

The converse direction can also be proved by converting the DNF formula E into the logically equivalent CNF formula D. □

We define two kinds of minimality for elementary/disjunctive explanations as follows.

Definition 3.3 A (disjunctive) explanation E of G is *minimal* if E is a *weakest* explanation of G, that is, if for any (disjunctive) explanation E' of G, $E \models E'$ implies $E' \models E$.

An explanation E of G is *least presumptive* if E is a minimal element among the explanations of G according to the less presumptive relation, that is, if for any explanation E' of G, $K \cup E \models E'$ implies $K \cup E' \models E$.

Note that we assumed that the set of abducibles $\mathcal{A}$ consists of literals only. Then, for elementary explanations E and E', the relation $E \models E'$ is equivalent to $E' \subseteq E$. Hence, E is a minimal elementary explanation of G iff no other explanation of G is a proper subset of E.

Definition 3.4 Let $\mathcal{ME}(G)$ be the set of minimal elementary explanations of G. The *explanation closure of G (wrt $\langle K, \mathcal{A} \rangle$)* is the disjunctive explanation:

$$\bigvee_{E \in \mathcal{ME}(G)} E.$$

The explanation closure gives us the least presumptive explanation for the observation. To verify this fact, we consider again an abductive program $\langle K, \mathcal{D}(\mathcal{A}) \rangle$ with the enlarged abducible set. Here, it should be noted that a weakest elementary explanation wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$ may contain redundant abducibles as its disjuncts. For instance, suppose that K is a program consisting of two rules:

$$\begin{aligned} p &\leftarrow a, \\ &\leftarrow b, \end{aligned}$$

and $\mathcal{A} = \{a, b\}$. Then, as p's explanations, $\{a; b\}$ is weaker than $\{a\}$. The notion of least presumptive explanations is necessary to adopt $\{a\}$ as a preferred explanation of p. In this case, $\{a\}$ and $\{a; b\}$ are equivalent relative to K.

Theorem 3.3 *If a formula F is the explanation closure of G wrt $\langle K, \mathcal{A} \rangle$, then F is equivalent (relative to K) to a least presumptive elementary explanation of G wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$. Conversely, if D is a least presumptive elementary explanation of G wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$, then D is equivalent (relative to K) to the explanation closure of G wrt $\langle K, \mathcal{A} \rangle$.*

Proof: Suppose the explanation closure F of G wrt $\langle K, \mathcal{A} \rangle$. By Proposition 3.2, there exists a CNF formula D which is equivalent to F relative to K such that D is an elementary explanation of G wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$. To show that D is least presumptive, suppose that D' is an elementary explanation of G wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$ such that $K \cup D \models K \cup D'$. By Proposition 3.2, there exists a DNF formula F' which is equivalent to D' relative to K such that F' is a disjunctive explanation of G wrt $\langle K, \mathcal{A} \rangle$. Then, $K \cup \{F\} \models K \cup \{F'\}$.

Let S' be any answer set of $K \cup \{F'\}$. Suppose that $S \not\subseteq S'$ holds for every answer set $S \in \mathcal{AS}(K \cup \{F\})$. This means that $S \setminus S' \neq \emptyset$ for any $S \in \mathcal{AS}(K \cup \{F\})$. In particular, one such an S satisfies that $S' \subset S$. This is because there is an answer set $S'' \in \mathcal{AS}(K \cup \{F'\})$ such that $S'' \subseteq S$ (by $K \cup \{F\} \models K \cup \{F'\}$), but by the minimality of answer sets we can put S'' as S' here. By the way, F' is a disjunctive explanation of G wrt $\langle K, \mathcal{A} \rangle$, then S' is also an answer set of $K \cup e'$ for some disjunct e' in F'. This e' is a non-minimal elementary explanation of G wrt $\langle K, \mathcal{A} \rangle$, because F contains all the minimal elementary explanations of G. Then, there is a minimal explanation e of G such that $e \subset e'$ and S is an answer set of $K \cup e$. However, $e' \subseteq S'$ and $S' \subset S$ imply $e' \subset S$. Thus S is also an answer set of $K \cup e'$. Then, both S and S' are answer sets of $K \cup e'$, but this is impossible by the minimality of answer sets, hence a contradiction. Therefore, there exist an answer set $S \in \mathcal{AS}(K \cup \{F\})$ such that $S \subseteq S'$. Hence, $K \cup \{F'\} \models K \cup \{F\}$. This implies that D is a least presumptive elementary explanation of G wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$.

The converse direction can also be proved in a similar way. □

Corollary 3.4 *The least presumptive elementary explanation of G wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$ is unique up to the equivalence relation relative to K, and is equivalent (relative to K) to the explanation closure of G wrt $\langle K, \mathcal{A} \rangle$.*

Proof: In Theorem 3.3, we saw that any least presumptive elementary explanation of G wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$ is equivalent (relative to K) to the explanation closure of G wrt $\langle K, \mathcal{A} \rangle$. Hence, all least presumptive elementary explanations of G wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$ are equivalent relative to K. □

Example 3.1 Let K be the program:

$$\begin{aligned} p; \neg q &\leftarrow a,\ b, \\ p &\leftarrow r,\ b, \\ q &\leftarrow c,\ not\, r, \\ r &\leftarrow d,\ not\, q. \end{aligned}$$

Also let the abducibles be $\mathcal{A} = \{a, b, c, d\}$. Then, the minimal elementary explanations of p wrt $\langle K, \mathcal{A} \rangle$ is: $\mathcal{ME}(p) = \{\{a, b, c\}, \{b, d\}\}$. The explanation closure of p is thus

$$F = (a, b, c); (b, d).$$

On the other hand, the least presumptive elementary explanation of p wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$ is given by

$$E = \{a; d,\ \ b,\ \ c; d\}.$$

In fact, $\mathcal{AS}(K \cup E) = \mathcal{AS}(K \cup \{F\}) = \{\{a, b, c, p, q\}, \{b, d, p, r\}\}$.

We now show that the explanation closure F of G wrt $\langle K, \mathcal{A} \rangle$ exactly reflects all the possible *minimal changes* from the original program K with the minimal elementary explanations $\mathcal{ME}(G)$ wrt $\langle K, \mathcal{A} \rangle$. With this property, we argue that all possible explanations can be *assimilated* into the current program so that the resulting program $K \cup \{F\}$ is uniquely determined. Note here that F is a DNF formula. If necessary, we can convert F into an equivalent CNF formula (see Theorem 3.3) which is in the form of a program. The merit of introduction of explanation closures is that we can just stay in the traditional abductive framework where the abducibles are given as literals and hence it is not necessary to consider the enlarged

abducible set $\mathcal{D}(\mathcal{A})$ for computing least presumptive explanations. Note that the computational complexity of explanation closures is the same as that of computing the minimal elementary explanations wrt $\langle K, \mathcal{A} \rangle$, which is exponential even in the case of an NLP without NAF (i.e, a definite Horn program).

In the following, for a set $\mathcal{S}$ of literal sets, we denote the set of minimal elements in $\mathcal{S}$ as $\mu\,\mathcal{S}$, i.e.,

$$\mu\,\mathcal{S} = \{\, I \in \mathcal{S} \mid \text{there is no } J \in \mathcal{S} \text{ such that } J \subset I \,\}.$$

Lemma 3.5 *Suppose an abductive program $\langle K, \mathcal{A} \rangle$, and let $\mathcal{M}$ be a set of abducible sets, i.e., $\mathcal{M} \subseteq 2^{\mathcal{A}}$. Then,*

$$\mathcal{AS}(K \cup \{ \bigvee_{e \in \mathcal{M}} e \}) = \mu \bigcup_{e \in \mathcal{M}} \mathcal{AS}(K \cup e).$$

Proof: Let α be $\mu \bigcup_{e \in \mathcal{M}} \mathcal{AS}(K \cup e)$. Suppose that $\mathcal{M} = \{e_1, \ldots, e_n\}$. Let S be any answer set of $K \cup \{e_1; \cdots; e_n\}$. Then, $e_i \subseteq S$ for some $e_i \in \mathcal{M}$ by the definition of answer sets. This S is also an answer set of $K \cup e_i$. By the minimality of answer sets, $S \in \alpha$ holds.

Conversely, suppose that S is an answer set in α. Then, $S \in \mathcal{AS}(K \cup e_i)$ for some $e_i \in \mathcal{M}$. Because any rule from K having an abducible in its head is always a fact consisting of abducibles only by assumption, adding e_i to K may not change the truth value of a literal in $e_j \setminus (K \cup e_i)$ for any $e_j \in \mathcal{M}$ $(j \neq i)$. Since S is minimal in α, S is also an answer set of $K \cup \{e_1; \cdots; e_n\}$. Hence, the lemma holds. □

Theorem 3.6 *Let F be the explanation closure of G wrt $\langle K, \mathcal{A} \rangle$, and $\mathcal{ME}(G)$ be the set of minimal elementary explanations of G wrt $\langle K, \mathcal{A} \rangle$. Then,*

$$\mathcal{AS}(K \cup \{F\}) = \mu \bigcup_{E \in \mathcal{ME}(G)} \mathcal{AS}(K \cup E).$$

Proof: This theorem is a special case of $\mathcal{M} = \mathcal{ME}(G)$ in Lemma 3.5. □

Note in Theorem 3.6 that the program augmented with the explanation closure $K \cup \{F\}$ preserves *all and only minimal answer sets* from the collection of programs with individual minimal elementary explanations. In other words, non-minimal answer sets produced by the minimal elementary explanations together with K are lost in $\mathcal{AS}(K \cup \{F\})$. This is because the program $K \cup \{F\}$ is an EDP, of which any answer set is minimal. For example, when the program K is

$$\begin{array}{rcl} a; b & , & \\ p & \leftarrow & b, \\ p & \leftarrow & c, \end{array}$$

and $\mathcal{A} = \{a, b, c\}$ is the abducibles, we have $\mathcal{ME}(p) = \{b, c\}$. Then,

$$\mathcal{AS}(K \cup \{b\}) \cup \mathcal{AS}(K \cup \{c\}) = \{\{b, p\}, \{a, c, p\}, \{b, c, p\}\}.$$

On the other hand,

$$\mathcal{AS}(K \cup \{b; c\}) = \{\{b, p\}, \{a, c, p\}\}.$$

When we consider the skeptical entailment, non-minimal answer sets are not useful and eliminating them does not change the consequences that are true in all answer sets.

Lemma 3.5 can also be applied for the set of *best* explanations of an observation, in which the criteria for best explanations are given by some preference or priority relations or probability measures. Namely, the notion of disjunctive explanations is useful not only for multiple minimal or least presumptive explanations, but for any multiple preferred explanations.

4 Disjunctions in Extended Abduction

In this section, we extend the notion of disjunctive explanations to allow for removal of abducible disjunctions from a program.

We firstly give a definition for *extended abduction* [19, 21, 35, 15]. The following definition is based on [15].

Definition 4.1 Let $\langle K, \mathcal{A} \rangle$ be an abductive program.

1. A pair (P, N) is a *scenario for* $\langle K, \mathcal{A} \rangle$ if P and N are sets of ground instances of elements from $\mathcal{A}$ and $(K \setminus N) \cup P$ is a consistent program.
2. Let G be a ground literal.
   (a) A pair (P, N) is an *elementary explanation of* G *(wrt* $\langle K, \mathcal{A} \rangle$*)* if (P, N) is a scenario for $\langle K, \mathcal{A} \rangle$ such that $(K \setminus N) \cup P \models G$.
   (b) A pair (P, N) is an *elementary anti-explanation of* G *(wrt* $\langle K, \mathcal{A} \rangle$*)* if (P, N) is a scenario for $\langle K, \mathcal{A} \rangle$ such that $(K \setminus N) \cup P \not\models G$.
   (c) An elementary (anti-)explanation (P, N) of G is *minimal* if for any elementary (anti-)explanation (P', N') of G, $P' \subseteq P$ and $N' \subseteq N$ imply $P' = P$ and $N' = N$.

Thus, to explain or unexplain observations, extended abduction not only introduces hypotheses to a program but also removes them from it. On the other hand, abduction in Definition 3.1 is called *normal abduction*, which only introduces hypotheses to explain observations, and is a special case of extended abduction. The next property obviously holds from the definitions of normal and extended abduction.

Proposition 4.1 *Let* $\langle K, \mathcal{A} \rangle$ *be an abductive program, and* G *an observation.* E *is an explanation of* G *wrt* $\langle K, \mathcal{A} \rangle$ *under normal abduction iff* $(E, \emptyset)$ *is an explanation of* G *wrt* $\langle K, \mathcal{A} \rangle$ *under extended abduction.*

For removing abducibles containing variables from a program, the set difference operation is semantically defined on ground programs as $S \setminus T = ground(S) \setminus ground(T)$, where $ground(S)$ is the ground instances of elements from S. Also, any literal/rule in a set is identified with its variant modulo variable renaming. For example, when $p(x) \in K$, for any constant a in the language of K, $\{p(a)\} \setminus K = \emptyset$ and $K \setminus \{p(a)\} = (K \setminus \{p(x)\}) \cup \{p(y) \mid y \neq a\}$.

4.1 Problem in Combining Removed Hypotheses

It is not obvious to extend the notion of elementary (anti-)explanations in extended abduction to take disjunctions of multiple (anti-)explanations. The difficulty lies in the following question: when there are more than one way to remove hypotheses in order to (un)explain an observation, how can we construct a combined (anti-)explanations so that the resulting program reflects the semantics for every possible minimal change of the current program? We illustrate this difficulty with the following example.

Example 4.1 [12, Example 3.4] [6] Let K be the program

$$\begin{array}{rcl} p & \leftarrow & a,\ b, \\ p & \leftarrow & e, \\ p & \leftarrow & q,\ c, \\ q & \leftarrow & a,\ d, \\ a & , & \\ b; d & , & \\ b; e & . & \end{array}$$

Suppose that the abducibles are $\mathcal{A} = \{a, b, c, d, e\}$. The unique minimal elementary anti-explanations of p wrt $\langle\, K, \mathcal{A}\,\rangle$ is

$$(P_1, N_1) = (\emptyset, \{a\}).$$

On the other hand, there are two minimal elementary anti-explanations of p wrt $\langle\, K, \mathcal{D}(\mathcal{A})\,\rangle$: one is (P_1, N_1), and the other is

$$(P_2, N_2) = (\emptyset, \{b; e\}).$$

To express these two changes in one state, Grant *et al.* [12] actually construct the two programs by reflecting these two anti-explanations on the fact part $\mathcal{F}(K)$:

$$\begin{array}{rcl} K_1 & = & \mathcal{I}(K) \cup \{\, b; d,\ \ b; e\,\}, \\ K_2 & = & \mathcal{I}(K) \cup \{\, a,\ \ b; d\,\}. \end{array}$$

Then, [12] takes the disjunction of these fact parts, i.e., $\mathcal{F}(K_1) \vee \mathcal{F}(K_2)$, and converting the resulting DNF formula into CNF, yielding

$$((b \vee d) \wedge (b \vee e)) \vee (a \wedge (b \vee d)) = (b \vee d) \wedge (a \vee b \vee e).$$

That is, the new program is computed as

$$K' = \mathcal{I}(K) \cup \{\, b; d,\ \ a; b; e\,\}.$$

By computing the difference between K and K', an anti-explanation of p would be expressed as

$$(P', N') = (\{\, a; b; e\,\},\ \{\, a,\ b; e\,\}).$$

Unless we follow this expensive procedure, it is difficult to compose the last scenario (P', N') directly from the minimal elementary anti-explanations, (P_1, N_1) and (P_2, N_2), of p wrt $\langle\, K, \mathcal{D}(\mathcal{A})\,\rangle$. Moreover, it is impossible to construct (P', N') only from the unique minimal elementary explanation (P_1, N_1) of p wrt $\langle\, K, \mathcal{A}\,\rangle$.

From the above example, one may expect that two (anti-)explanations, (P_1, N_1) and (P_2, N_2), can be combined by constructing a new (anti-)explanation:

$$(\{P_1 \vee P_2,\ N_1 \vee N_2\},\ N_1 \cup N_2).$$

Unfortunately, this is not the case as the next example shows.

[6] Example 4.1 was originally described in the context of *view updates* of disjunctive databases in [12]. Here, we modified it for the use in extended abduction.

Example 4.2 Let K be the program

$$p \leftarrow a,\ not\ b,$$
$$p \leftarrow a,\ not\ c,$$
$$b,$$
$$c,$$

and the abducibles be $\mathcal{A} = \{a, b, c\}$. The two minimal elementary explanations of p are $(\{a\}, \{b\})$ and $(\{a\}, \{c\})$. Combining these two in the above way results in $(P, N) = (\{\, a,\ b; c\,\},\ \{\, b,\ c\,\})$. However, this scenario cannot be an explanation of p because $(K \setminus N) \cup P \not\models p$.

4.2 From Extended Abduction to Normal Abduction

From the discussion in Section 4.1, we had better consider an alternative way to combine multiple (anti-)explanations in extended abduction. In [15], extended abduction is shown to be reduced to normal abduction. Here, we use this method to translate removal of abducibles from a program into addition of abducibles to a program. Recall that, without loss of generality, the set of abducibles $\mathcal{A}$ can be assumed to be a set of literals and there is no rule which has a non-empty body and a head containing abducible literals. Under this assumption, the translation ν shown in [15] is simplified as follows. For addition of an abducible literal, we do not have to give it a name and leave it as it is. For removal of an abducible literal a, we give a name to a through NAF by $not\ del(a)$. Then, deletion of an abducible a is realized by addition of $del(a)$ to the program.

For an abductive program $\langle\, K, \mathcal{A}\,\rangle$, the program $\nu(K, \mathcal{A}) = \langle\, \nu(K), \nu(\mathcal{A})\,\rangle$ is defined as follows.

$$\begin{aligned} \nu(K) &= (K \setminus \mathcal{A}) \cup \{\, a \leftarrow not\ del(a) \mid a \in K \cap \mathcal{A}\,\}, \\ \nu(\mathcal{A}) &= \mathcal{A} \cup \{\, del(a) \mid a \in K \cap \mathcal{A}\,\}. \end{aligned}$$

Theorem 4.2 [15, Theorem 1][7] *If (P, N) is a minimal elementary explanation of G wrt $\langle\, K, \mathcal{A}\,\rangle$ under extended abduction, then E is a minimal elementary explanation of G wrt $\nu(K, \mathcal{A})$ under normal abduction, where $E = P \cup \{del(a) \mid a \in N\}$.*

Conversely, if E is a minimal elementary explanation of G wrt $\nu(K, \mathcal{A})$ under normal abduction, then (P, N) is a minimal elementary explanation of G wrt $\langle\, K, \mathcal{A}\,\rangle$ under extended abduction, where $P = E \cap \mathcal{A}$ and $N = \{a \mid del(a) \in E\}$.

The above theorem presents that all minimal elementary explanations are computable by normal abduction from $\nu(K, \mathcal{A})$. As argued in [15], the translation from extended abduction to normal abduction enables us to derive the computational complexity of extended abduction, which is the same as that of normal abduction.

For anti-explanations, the next theorem shows that $\nu(K, \mathcal{A})$ preserves every minimal elementary anti-explanation of $\langle\, K, \mathcal{A}\,\rangle$ in the form of a scenario $(E, \emptyset)$. Namely, we do not have to consider removal of hypotheses in a scenario. Then, to compute these anti-explanations, we can utilize the relationship between explanations and anti-explanations (see [15, Theorem 2]):

[7] The complete proof of Theorem 1 is not included in [15]. For Theorem 4.2 in this paper, the proof can be obtained by considering explanations instead of anti-explanations in the proof of Theorem 4.3.

(E, F) is an anti-explanation of G wrt $\langle K, \mathcal{A} \rangle$ iff (E, F) is a (credulous) explanation of G' wrt $\langle K \cup \{ G' \leftarrow not\ G \}, \mathcal{A} \rangle$.

Theorem 4.3 *If (P, N) is a minimal elementary anti-explanation of G wrt $\langle K, \mathcal{A} \rangle$, then $(E, \emptyset)$ is a minimal elementary anti-explanation of G wrt $\nu(K, \mathcal{A})$, where $E = P \cup \{del(a) \mid a \in N\}$.*

Conversely, if $(E, \emptyset)$ is a minimal elementary anti-explanation of G wrt $\nu(K, \mathcal{A})$, then (P, N) is a minimal elementary anti-explanation of G wrt $\langle K, \mathcal{A} \rangle$, where $P = E \cap \mathcal{A}$ and $N = \{a \mid del(a) \in E\}$.

Proof: Suppose that (P, N) is a minimal elementary anti-explanation of G wrt $\langle K, \mathcal{A} \rangle$. Then, $(K \setminus N) \cup P \not\models G$. Since $(K \setminus N) \cup P$ is consistent, take its any answer set S. Let $S' = S \cup \{del(a) \mid a \in N\}$, and $E = P \cup \{del(a) \mid a \in N\}$. Then, $\nu(K)^{S'} = (K \setminus \mathcal{A})^S \cup ((K \cap \mathcal{A}) \setminus N) = (K \setminus N)^S$. By $E^{S'} = E$, $(\nu(K) \cup E)^{S'} = (K \setminus N)^S \cup E = ((K \setminus N) \cup P)^S \cup \{del(a) \mid a \in N\}$. Since S is an answer set of $((K \setminus N) \cup P)^S$, S' is an answer set of $(\nu(K) \cup E)^{S'}$. Thus, S' is an answer set of $\nu(K) \cup E = (\nu(K) \setminus \emptyset) \cup E$. Obviously, $G \notin S'$ iff $G \notin S$. Hence, $(E, \emptyset)$ is an elementary anti-explanation of G wrt $\nu(K, \mathcal{A})$. The minimality of E is assured by the minimality of (P, N).

The converse direction can also be proved by tracing the above proof backward. □

4.3 Disjunctive (Anti-)Explanations

Now, we are ready to compose disjunctive explanations for extended abduction. Firstly, we extend Definition 4.1 for extended abduction by allowing removal of disjunctive hypotheses from a program.

Definition 4.2 Let $\langle K, \mathcal{A} \rangle$ be an abductive program, G a ground literal.

1. A pair (P, N) is a *d-scenario for* $\langle K, \mathcal{A} \rangle$ if P is a set of ground instances of elements from $\mathcal{A}$ and N is a set of ground instances of elements from $\mathcal{D}(\mathcal{A})$ such that $(K \setminus N) \cup P$ is a consistent.
2. A d-scenario (P, N) is an *elementary d-explanation of G (wrt $\langle K, \mathcal{A} \rangle$)* if $(K \setminus N) \cup P \models G$.
3. A d-scenario (P, N) is an *elementary d-anti-explanation of G (wrt $\langle K, \mathcal{A} \rangle$)* if $(K \setminus N) \cup P \not\models G$.
4. An elementary d-(anti-)explanation (P, N) of G is *minimal* if for any elementary d-(anti-)explanation (P', N') of G, $P \models P'$ and $N \models N'$ imply $P' \models P$ and $N' \models N$.

In the above definition, we allow removal of disjunctive hypotheses from the enlarged abducible set $\mathcal{D}(\mathcal{A})$, but addition of hypotheses is allowed only from the literal abducibles $\mathcal{A}$. This asymmetry is due to our intention that hypotheses to be added should be made disjunctive just in the same way as normal abduction although hypotheses to be removed could only be translated into normal abduction through NAF of the form *not* $del(_)$. Note also that the minimality of d-(anti-)explanations is now defined through the entailment relationship.

For translating abducible removal into abducible addition, we slightly modify the mapping ν for preserving minimal elementary (anti-)explanations, and consider the mapping ν^d as follows. For an abductive program $\langle K, \mathcal{A} \rangle$, the program $\nu^d(K, \mathcal{A}) = \langle \nu^d(K), \nu^d(\mathcal{A}) \rangle$ is defined as follows.

$$\begin{aligned} \nu^d(K) &= (K \setminus \mathcal{D}(\mathcal{A})) \cup \{ a \leftarrow not\ del(a) \mid a \in K \cap \mathcal{D}(\mathcal{A}) \}, \\ \nu^d(\mathcal{A}) &= \mathcal{A} \cup \{ del(a) \mid a \in K \cap \mathcal{D}(\mathcal{A}) \}. \end{aligned}$$

Note that the difference between ν and ν^d is that the naming technique is applied to the enlarged abducible set $\mathcal{D}(\mathcal{A})$ instead of the original abducibles $\mathcal{A}$ only. The new abducible set $\nu^d(\mathcal{A})$ is, however, defined with $\mathcal{A}$ without considering disjunctive hypotheses. This is because we do not have to consider any removal of hypotheses for $\nu^d(K, \mathcal{A})$, so that we can define the notions of *(disjunctive) explanations* and *minimal explanations* in the same way as Definitions 3.2 and 3.3 for normal abduction.

The next two theorems are direct extensions of Theorems 4.2 and 4.3, hence we omit their proofs.

Theorem 4.4 *If (P, N) is a minimal elementary d-explanation of G wrt $\langle K, \mathcal{A} \rangle$, then E is a minimal elementary explanation of G wrt $\nu^d(K, \mathcal{A})$, where $E = P \cup \{del(d) \mid d \in N\}$.*

Conversely, if E is a minimal elementary explanation of G wrt $\nu^d(K, \mathcal{A})$, then (P, N) is a minimal elementary d-explanation of G wrt $\langle K, \mathcal{A} \rangle$, where $P = E \cap \mathcal{A}$ and $N = \{d \mid del(d) \in E\}$.

Theorem 4.5 *If (P, N) is a minimal elementary d-anti-explanation of G wrt $\langle K, \mathcal{A} \rangle$, then $(E, \emptyset)$ is a minimal elementary anti-explanation of G wrt $\nu^d(K, \mathcal{A})$, where $E = P \cup \{del(d) \mid d \in N\}$.*

Conversely, if $(E, \emptyset)$ is a minimal elementary anti-explanation of G wrt $\nu^d(K, \mathcal{A})$, then (P, N) is a minimal elementary d-anti-explanation of G wrt $\langle K, \mathcal{A} \rangle$, where $P = E \cap \mathcal{A}$ and $N = \{d \mid del(d) \in E\}$.

Like Definition 3.4 for normal abduction, we can define the closure formula for (anti-)explanations in extended abduction as follows.

Definition 4.3 The *explanation closure of G (wrt $\langle K, \mathcal{A} \rangle$)* (for extended abduction) is the disjunctive explanation:

$$\bigvee_{E \in \mathcal{ME}^{\nu}(G)} E,$$

where $\mathcal{ME}^{\nu}(G)$ is the set of all minimal elementary explanations of G wrt $\nu^d(K, \mathcal{A})$.

Definition 4.4 The *anti-explanation closure of G (wrt $\langle K, \mathcal{A} \rangle$)* is the disjunctive explanation:

$$\bigvee_{(E,\emptyset) \in \mathcal{MA}^{\nu}(G)} E,$$

where $\mathcal{MA}^{\nu}(G)$ is the set of all minimal elementary anti-explanations of G wrt $\nu^d(K, \mathcal{A})$.

The following theorems show that the translation ν^d preserves the minimal answer sets from the program augmented with any minimal elementary d(-anti)-explanation. Here, for a program K containing literals of the form $del(_)$, we will write:

$$\mathcal{AS}^{-del}(K) = \mu \{ S \cap \mathcal{L}_K \mid S \in \mathcal{AS}(K) \},$$

where $\mathcal{L}_K$ denotes the set of literals in the language of K not containing any literal of the form $del(_)$. Note that we need to select the minimal elements from the right hand side. This is because eliminating all literals of the form $del(_)$ from each answer set may produce a literal set that properly includes others.

Theorem 4.6 *Let F be the explanation closure of G wrt $\langle K, \mathcal{A} \rangle$, and $\mathcal{ME}^d(G)$ be the set of minimal elementary d-explanations of G wrt $\langle K, \mathcal{A} \rangle$. Then,*

$$\mathcal{AS}^{-del}(\nu^d(K) \cup \{F\}) = \mu \bigcup_{(P,N) \in \mathcal{ME}^d(G)} \mathcal{AS}((K \setminus N) \cup P).$$

Proof: Let $\mathcal{ME}^\nu(G)$ be the set of all minimal elementary explanations of G wrt $\nu^d(K, \mathcal{A})$. Then, by Theorem 3.6, it holds that

$$\mathcal{AS}(\nu^d(K) \cup \{F\}) = \mu \bigcup_{E \in \mathcal{ME}^\nu(G)} \mathcal{AS}(\nu^d(K) \cup E).$$

In the right-hand side of this equation, by Theorem 4.4, for each $E \in \mathcal{ME}^\nu(G)$, there exists $(P, N) \in \mathcal{ME}^d(G)$ such that $P = E \cap \mathcal{A}$ and $N = \{d \mid del(d) \in E\}$. On the other side, for each answer set $T \in \mathcal{AS}(\nu^d(K) \cup \{F\})$, $T \cap \mathcal{L}_K = T \setminus \{del(d) \mid d \in T\}$ holds. Then, deleting every literal of the form $del(_)$ from each answer set in both sides, we get the theorem. □

Theorem 4.7 *Let H be the anti-explanation closure of G wrt $\langle K, \mathcal{A} \rangle$, and $\mathcal{MA}^d(G)$ be the set of minimal elementary d-anti-explanations of G wrt $\langle K, \mathcal{A} \rangle$. Then,*

$$\mathcal{AS}^{-del}(\nu^d(K) \cup \{H\}) = \mu \bigcup_{(P,N) \in \mathcal{MA}^d(G)} \mathcal{AS}((K \setminus N) \cup P).$$

Proof: The proof of this theorem is similar to that of Theorem 4.6. Instead of the equation derived by Theorem 3.6, we have a similar result:

$$\mathcal{AS}(\nu^d(K) \cup \{H\}) = \mu \bigcup_{(E,\emptyset) \in \mathcal{MA}^\nu(G)} \mathcal{AS}(\nu^d(K) \cup E),$$

where $\mathcal{MA}^\nu(G)$ is the set of all minimal elementary anti-explanations of G wrt $\nu^d(K, \mathcal{A})$. For the rest of the proof, Theorem 4.5 is used instead of Theorem 4.4. □

Example 4.3 (cont. from Example 4.1) The fact part $\mathcal{I}(K) = K \cap \mathcal{D}(\mathcal{A}) = \{\, a,\ b; d,\ b; e \,\}$ is translated into

$$\begin{aligned} a &\leftarrow not\ del(a), \\ b; d &\leftarrow not\ del(b; d), \\ b; e &\leftarrow not\ del(b; e). \end{aligned}$$

The two minimal elementary anti-explanations of p wrt $\nu^d(K, \mathcal{A})$ are $(E'_1, \emptyset)$ and $(E'_2, \emptyset)$, where

$$E'_1 = \{del(a)\} \text{ and } E'_2 = \{del(b; e)\},$$

which respectively correspond to the two d-anti-explanations of p wrt $\langle K, \mathcal{A} \rangle$, $(P_1, N_1) = (\emptyset, \{a\})$ and $(P_2, N_2) = (\emptyset, \{b; e\})$. Then, the anti-explanation closure of p is $H = del(a); del(b; e)$. Assimilating this formula into the program, we obtain the new program

$$K' = \nu^d(K) \cup \{del(a); del(b; e)\},$$

whose answer sets are

$$\mathcal{AS}(K') = \{\{b, del(a)\}, \{d, e, p, del(a)\}, \{a, b, p, del(b; e)\}, \{a, d, q, del(b; e)\}\}.$$

Then,

$$\mathcal{AS}^{-del}(K') = \{\{b\}, \{d, e, p\}, \{a, d, q\}\}.$$

Example 4.4 (cont. from Example 4.2) The fact part $\mathcal{F}(K)$ is translated into

$$\begin{aligned} b &\leftarrow \mathit{not}\, del(b), \\ c &\leftarrow \mathit{not}\, del(c). \end{aligned}$$

The two minimal elementary explanations of p wrt $\nu^d(K, \mathcal{A})$ are $\{a, del(b)\}$ and $\{a, del(c)\}$, which respectively correspond to $(\{a\}, \{b\})$ and $(\{a\}, \{c\})$. Then, the explanation closure is $F = (a, del(b)); (a, del(c))$. By converting F into CNF, the minimal explanation of p wrt $\nu^d(K, \mathcal{A})$ is obtained as $E = \{\, a,\ del(b); del(c)\,\}$. Assimilating this explanation into $\nu^d(K)$ yields the answer sets:

$$\mathcal{AS}^{-del}(\nu^d(K) \cup E) = \{\{a, b, p\}, \{a, c, p\}\}.$$

5 Related Work

1. Disjunctive explanations. The idea of taking a disjunction of multiple explanations has appeared at times in the literature of computing abduction, although no previous work has formally investigated the effect of such disjunctive explanations in depth. Helft *et al.* [13] define an explanation as a disjunction of elementary explanations in abduction from first-order theories for answering queries in *circumscription*. Konolige [26] defines a *cautious explanation* as a disjunction of all preferred explanations, and uses it to relate consistency-based explanations with abductive explanations in propositional causal theories. Lin [29] provides a method to compute *weakest sufficient conditions* for propositional theories, in which he constructs the disjunction of elementary explanations obtained from prime implicates. In ALP, disjunctions of elementary explanations are sometimes obtained in computing abduction through Clark completion [3, 10, 30]. Such procedures are designed for computing normal abduction from hierarchical or acyclic NLPs. Inoue and Sakama [21] extend this completion method to compute extended abduction. We can use these procedures to compute explanation closures directly in some restricted classes of logic programs.

2. View updates in disjunctive databases. Although there are some studies on updating incomplete information in relational databases [1], only a few works [12, 9] focused on updating disjunctive databases. Grant *et al.* [12] translate view updates into a set of disjunctive facts based on expansion of an SLD-tree, so that updates are achieved by inserting/deleting these disjunctive facts to/from a database. Their method is correct for stratified programs, but cannot achieve an insertion of p into a non-stratified EDP K shown in Example 3.1. Fernández *et al.* [9] realize view updates in a wide class of EDPs through construction of minimal models that satisfy an update request. In their algorithm, however, computation is done on all possible models of the Herbrand base, and how to compute disjunctive solutions directly from changes of facts was an *open problem* in the class of EDPs. We solved this problem by translating extended abduction to normal abduction without computing all possible models. Furthermore, updates are performed without using abduction in [12, 9]. Hence, the notion of disjunctive (anti-)explanations in abduction does not appear in these work.

For non-disjunctive deductive databases, abductive frameworks have been used to realize view updates. Bry [2] translates abduction from an NLP into a disjunctive program and database updates are realized by bottom-up

computation on a meta-program specifying an update procedure. Kakas and Mancarella [24] characterize view updates through abduction in deductive databases represented in NLPs. The procedures in [24, 2] are based on normal abduction from NLPs, and do not consider extended abduction from disjunctive programs.

3. Knowledge assimilation with abduction. Not much work has been reported to assimilate obtained multiple explanations into the current knowledge base. Kakas and Mancarella [25] discussed two ways for handling the problem of multiple explanations. One is to generate all consistent scenarios accounting for an observation and work with all of them simultaneously. They suggest to use an ATMS [4] for this purpose. The other way is to generate one preferred explanation at a time according to some priority. Since such a choice of explanation could be wrong in the subsequent observations, they suggest the use of a belief revision mechanism through a Doyle-style TMS [5].

Our proposal somewhat differs from Kakas and Mancarella's two methods. Our method is similar to the spirit suggested by Fagin *et al.* [7], which defines the result of assimilation or updates to be the disjunction of all the possible theories with minimal change. This method presents a semantically consistent picture of theory changes. Rossi and Naqvi [34] optimize this approach by taking the disjunction of updated extensional databases instead of composing the disjunction of the whole databases with intensional ones. Grant *et al.* [12] follow the same line on view updates in disjunctive databases. An interesting alternative approach is also suggested by Fagin *et al.* [8], in which multiple alternative theories called "flocks" are kept as they are.

6 Conclusion

This paper has presented several properties of disjunctive explanations in the context of (abductive) logic programming. The contribution of this paper can be summerized as follows.

First, we have shown a method to construct the least presumptive explanations and anti-explanations in normal and extended abduction. For normal abduction, we formally established the effect of disjunctive explanations, in which all and only minimal answer sets are preserved for the minimal elementary explanations. We also presented that the explanation closure is equivalent to a least presumptive explanation consisting of disjunctive hypotheses.

Second, we have investigated a new framework of abductive logic programming which allows disjunctive hypotheses as abducibles. Abduction from $\langle K, \mathcal{D}(\mathcal{A}) \rangle$ is worth considering, as it can be applied to knowledge assimilation, view updates, skeptical reasoning, specialization in ILP, and so on. The results in this paper imply a practical merit that computing least presumptive explanations wrt $\langle K, \mathcal{D}(\mathcal{A}) \rangle$ can easily be realized by traditional abductive procedures [24, 3, 20, 10, 23, 21] for $\langle K, \mathcal{A} \rangle$. In other words, abduction from $\langle K, \mathcal{D}(\mathcal{A}) \rangle$ can be implemented by considering disjunctive explanations which are constructed with elementary explanations in abduction from $\langle K, \mathcal{A} \rangle$. That is, after the minimal elementary explanations are computed by these procedures, the disjunction of them is just composed. Alternatively, we can use *answer set programming* [31], in which normal abduction for elementary explanations is simulated in computing

answer sets [36, 14, 20]. We have also applied these results to extended abduction from $\langle K, \mathcal{D}(\mathcal{A}) \rangle$, and proposed a method to combine multiple solutions that involve removal of hypotheses. Extended abduction in this case is converted into normal abduction.

Third, the notion of disjunctive explanations is quite useful in various applications, and our method has shed some light on the problem of knowledge assimilation. In particular, considering view updates in disjunctive databases is generally difficult in the presence of disjunctive information. Our solution in this paper correctly achieves view updates in a large class of disjunctive databases.

In future work, we plan to incorporate the notion of disjunctive explanations in an ILP system, and formalize and implement induction via abduction. Since we have constructed DNF/CNF formulas as disjunctive explanations in this paper, such techniques may be well applied to construct clausal theories that account for positive and negative evidences. Then, the notion of minimal or least presumptive explanations in abduction could be applied to select better hypotheses in ILP.

References

[1] S. Abiteboul. Updates, a new frontier. In: *Proceedings of the 2nd International Conference on Database Theory, Lecture Notes in Computer Science*, 326, Springer, pages 1–18, 1988.

[2] F. Bry. Intensional updates: abduction via deduction. In: *Proceedings of the 7th International Conference on Logic Programming*, pages 561–575, MIT Press, 1990.

[3] L. Console, D. Theseider Dupré and P. Torasso. On the relationship between abduction and deduction. *Journal of Logic and Computation*, 1:661–690, 1991.

[4] J. de Kleer. An assumption-based TMS. *Artificial Intelligence*, 28:127–162, 1986.

[5] J. Doyle. A truth maintenance system. *Artificial Intelligence*, 12:231–272, 1979.

[6] T. Eiter, G. Gottlob, and H. Mannila. Adding disjunction to Datalog. In: *Proceedings of the 13th ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems*, pages 267–278, 1994.

[7] R. Fagin, J. D. Ullman, and M. Y. Vardi. On the semantics of updates in databases (preliminary report). In: *Proceedings of the 2nd ACM SIGACT-SIGMOD Symposium on Principles of Database Systems*, pages 352–365, 1983.

[8] R. Fagin, G. M. Kuper, J. D. Ullman, and M. Y. Vardi. Updating logical databases. In: *Advances in Computing Research*, Volume 3, pages 1–18, JAI Press, 1986.

[9] J. Fernández, J. Grant and J. Minker. Model theoretic approach to view updates in deductive databases. *Journal of Automated Reasoning*, 17:171–197, 1996.

[10] T. H. Fung and R. Kowalski. The iff procedure for abductive logic programming. *Journal of Logic Programming*, 33:151–165, 1997.

[11] M. Gelfond and V. Lifschitz. Classical negation in logic programs and disjunctive databases. *New Generation Computing*, 9:365–385, 1991.

[12] J. Grant, J. Horty, J. Lobo and J. Minker. View updates in stratified disjunctive databases. *Journal of Automated Reasoning*, 11:249–267, 1993.

[13] N. Helft, K. Inoue, and D. Poole. Query answering in circumscription. In: *Proceedings of the 12th International Joint Conference on Artificial Intelligence* (IJCAI-91), pages 426–431, Morgan Kaufmann, 1991.

[14] K. Inoue. Hypothetical reasoning in logic programs. *Journal of Logic Programming*, 18(3):191-227, 1994.

[15] K. Inoue. A simple characterization of extended abduction. In *Proceedings of the 1st International Conference on Computational Logic, Lecture Notes in Artificial Intelligence*, 1861, pages 718–732, Springer, 2000.

[16] K. Inoue. Induction, abduction, and consequence-finding. In: C. Rouveirol and M. Sebag (eds.), *Proceedings of the 11th International Conference on Inductive Logic Programming, Lecture Notes in Artificial Intelligence*, 2157, pages 65–79, Springer, 2001.

[17] K. Inoue. Automated abduction. In: A. C. Kakas and F. Sadri (eds.), *Computational Logic: Logic Programming and Beyond —Essays in Honour of Robert A. Kowalski, Part II, Lecture Notes in Artificial Intelligence*, 2408, Springer, 2002.

[18] K. Inoue, M. Koshimura, and R. Hasegawa. Embedding negation as failure into a model generation theorem prover. In: *Proceedings of the 11th International Conference on Automated Deduction* (CADE-11), *Lecture Notes in Artificial Intelligence*, 607, pages 400–415, Springer, 1992.

[19] K. Inoue and C. Sakama. Abductive framework for nonmonotonic theory change. In: *Proceedings of the 14th International Joint Conference on Artificial Intelligence* (IJCAI-95), pages 204–210, Morgan Kaufmann, 1995. pages 204–210, Morgan Kaufmann, 1995.

[20] K. Inoue and C. Sakama. A fixpoint characterization of abductive logic programs. *Journal of Logic Programming*, 27(2):107–136, 1996.

[21] K. Inoue and C. Sakama. Computing extended abduction through transaction programs. *Annals of Mathematics and Artificial Intelligence*, 25(3,4):339-367, 1999.

[22] K. Inoue and C. Sakama. Disjunctive explanations. In: P. J. Stuckey (ed.), *Proceedings of the 18th International Conference on Logic Programming, Lecture Notes in Computer Science*, 2401, pages 317–332, Springer, 2002.

[23] A. C. Kakas, R. A. Kowalski and F. Toni. The role of abduction in logic programming. In: D. M. Gabbay, C. J. Hogger and J. A. Robinson (eds.), *Handbook of Logic in Artificial Intelligence and Logic Programming*, volume 5, pages 235–324, Oxford University Press, 1998.

[24] A. C. Kakas and P. Mancarella. Database updates through abduction. In: *Proceedings of the 16th International Conference on Very Large Databases*, pages 650–661, Morgan Kaufmann, 1990.

[25] A. C. Kakas and P. Mancarella. Knowledge assimilation and abduction. In: J. P. Martins and M. Reinfrank (eds.), *Truth Maintenance Systems, Lecture Notes in Artificial Intelligence*, 515, 54–70, Springer, 1991.

[26] K. Konolige. Abduction versus closure in causal theories. *Artificial Intelligence*, 53:255–272, 1992.

[27] V. Lifschitz, D. Pearce, and A. Valverde. Strongly equivalent logic programs. *ACM Transactions on Computational Logic*, 2:526–541, 2001.

[28] V. Lifschitz, L. R. Tang, and H. Turner. Nested expressions in logic programs. *Annals of Mathematics and Artificial Intelligence*, 25:369–389, 1999.

[29] F. Lin. On strongest necessary and weakest sufficient conditions. In: *Proceedings of the 7th International Conference on Principles of Knowledge Representation and Reasoning*, pages 167–175, Morgan Kaufmann, 2000.

[30] F. Lin and J.-H. You. Abductive logic programming: a new definition and an abductive procedure based on rewriting. In: *Proceedings of the 17th International Joint Conference on Artificial Intelligence* (IJCAI-01), pages 655–661, Morgan Kaufmann, 2001.

[31] V. W. Marek, and M. Truszczyński. Stable models and an alternative logic programming paradigm. In: K. R. Apt *et al.*, editors, *The Logic Programming Paradigm—A 25 Year Perspective*, pages 375–398, Springer, 1999.

[32] S. Muggleton. Inverse entailment and Progol. *New Generation Computing*, 13:245–286, 1995.

[33] S.-H. Nienhuys-Cheng and R. de Wolf. *Foundations of Inductive Logic Programming*. LNAI, 1228, Springer, 1997.

[34] F. Rossi and S. A. Naqvi. Contributions to the view update problem. In: *Proceedings of the 6th International Conference on Logic Programming*, pages 398–415, MIT Press, 1989.

[35] C. Sakama and K. Inoue. Updating extended logic programs through abduction. In: *Proceedings of the 5th International Conference on Logic Programming and Nonmonotonic Reasoning*, *Lecture Notes in Artificial Intelligence*, 1730, pages 147-161, Springer, 1999.

[36] K. Satoh and N. Iwayama. Computing abduction by using the TMS. In: *Proceedings of the 8th International Conference on Logic Programming*, pages 505–518, MIT Press, 1991.