A short note on nonmonotonic inferences induced by graded similarity

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Abstract

We explore which kinds of nonmonotonic inference relations naturally arise when using similarity-based implication and consistency measures to rank propositions à la Gärdenfors and Makinson. There is no surprising result, the main interest being to provide a new perspective to nonmonotonic reasoning from a field which has not been traditionally considered within the uncertainty formalisms, but which is indeed very close, at least to Possibility theory.

Introduction

Reasoning under uncertainty deals with statements like "if p is true then q is *likely* to be true", where p and q are propositions of a given formal language $\mathcal{L}$. Such a statement expresses that knowing p is true is not enough information to conclude the truth of q, but still it is a positive evidence about the possible truth of q. The strength of the evidence is usually given by some kind of measure $g : 2^{\mathcal{W}} \to [0,1]$ on the set of subsets of possible worlds or possible states of the system of our concern. Usual choices of measures g are probability functions, possibility / necessity functions, or plausibility / belief functions. Given such a measure g, the certainty degree of a proposition p being true is then evaluated by the measure $g([p])$ of the set of worlds $[p] = \{w \in W \mid w \models p\}$ where p is true.

In contrast to uncertainty reasoning, the kind of statements which are in the scope of similarity-based reasoning are of the form "if p is true then q is *close* to be true", in the sense that, even though q may be false, the fact of p being true entails that q is (semantically) close (or similar) to some other propositions which are true. Notice that the fact of q being close to be (or approximately) true has nothing to do with a problem of uncertainty, i.e. a problem of missing information crucial to know whether q is true or false (see e.g. (Dubois and Prade 1995)). The matter here is to take into account that some situations resemble more than another. This leads to deal with degrees of truth (the closer q is to truth, the more q is true) rather than to degrees of uncertainty.

In trying to formalize similarity-based reasoning, the first question one is faced with is how to model the notion of similarity. Despite some qualitative approaches, like (Lewis 1973) or (Williamson 1988), a very natural account of similarity is as a graded binary relation among objects. Of particular interest is the approach started out by Ruspini in (Ruspini 1991) in order to provide Fuzzy logic with a similarity-based semantics. To do so he proposed the introduction of a fuzzy binary relation $S : \mathcal{W} \times \mathcal{W} \to [0,1]$, mapping pairs of possible worlds into reals between 0 and 1. The aim of such similarity function was to capture a notion of semantical proximity between possible worlds, with value 1 corresponding to the identity of possible worlds and value 0 indicating that knowledge of propositions that are true in one possible world does not provide any indication about propositions that are true in the other.

It is worth noticing that there have been recently a number of attempts to push further this approach to come up with well-founded similarity reasoning formalisms (see e.g. (Dubois *et al.* 1997; Esteva *et al.* 1997)). Despite of the fact that all these approaches are monotonic, there is a straightforward way of relating nonmonotonic reasoning and similarity reasoning. Namely, some measures of proximity definable from a similarity function among worlds naturally induce preordering relations between worlds or propositions that, in turn, are the usual tool required f.i. by Gärdenfors and Makinson to characterize well-behaved nonmonotonic consequence relations. In (Rodriguez *et al.* 1996) there is a first step to formally establish close links between some similarity-based orderings and revision and update operators.

In this short note we explore which kinds of nonmonotonic inference relations naturally arise when using the so-called *implication* and *consistency* measures to rank propositions à la Gärdenfors and Makinson. There is no surprising result, the main interest is to provide a new perspective to nonmonotonic reasoning from a field which has not been traditionally considered within the uncertainty formalisms, but which is indeed very close, at least to Possibility theory. Our approach is founded over two ideas: a new kind of orderings between sentences that we call inclusion orderings and a new implication measure $L_S(p \mid K)$ which indicates the degree of what $\neg p$ implies $\neg K$. These two notions are combined to obtain a new form to define comparative entailments.

The paper is organized as follows. Section 2 and Section 3 contain some basic background knowledge about the notion of similarity-based models and about comparative entailment relations including the set of corresponding properties and representation theorems. Section 4 and 5 are devoted to present our approach to connect comparative entailments and similarity-based models. Section 6 is about the relationship between sphere systems and orderings of formulas and to show that the similarity-based orderings introduced in Sections 4 and 5 are captured by a new class of orderings from sphere systems. Finally, Section 7 contains some discussion about related and future works.

Similarity relations, measures and approximate entailment

Some requirements must be satisfied by a fuzzy relation $S : \mathcal{W} \otimes \mathcal{W}$ to cast a notion of similarity. For example, the similarity degree of any world with itself must be the highest value, $S(w, w) = 1$, that is, S must be *reflexive*. It is also natural to require S be *symmetric*, $S(w, w') = S(w', w)$. Sometimes a relaxed form of *transitivity*, expressed as the inequality $S(w, w") \geq S(w, w') \otimes S(w', w")$, is also required, where $\otimes$ is considered to be a continuous triangular norm (or t-norm[1]). Furthermore, we shall say that a similarity function is *separating* if $S(w, w') = 1$ iff $w = w'$.

Fuzzy relations satisfying reflexivity, symmetry and $\otimes$-transitivity are the generalization of the well-known notion of equivalence relation. Indeed, in such a case, it is clear that if $S(\mathcal{W} \times \mathcal{W}) = \{0, 1\}$, S is nothing but an equivalence relation. Moreover, if S is separating, the 1-cut of S (that is, $\{(w, w') \mid S(w, w') = 1\}$) is just the equality on W.

From now on, we shall consider a finitely generated propositional logical language $\mathcal{L}$ and $\mathcal{W}$ will stand for the set of its Boolean interpretations or set of worlds.

Given the available information (evidence) modeled by a proposition K and a similarity function S on $\mathcal{W}$, Ruspini proposes a pair of measures on propositions of $\mathcal{L}$ defined as

$$I_S(p \mid K) = \inf_{w_1 \models K} \sup_{w_2 \models p} S(w_1, w_2)$$

$$C_S(p \mid K) = \sup_{w_1 \models K} \sup_{w_2 \models p} S(w_1, w_2)$$

to measure in what degree the evidence K implies and is consistent with p respectively. I_S is a so-called *implication* measure and C_S is a *consistency* measure (Ruspini 1991). These measures, as Ruspini points out, are respectively lower and upper bounds of the resemblance degrees between p-worlds and K-worlds from the point of view of K.

The value of $I_S(p \mid K)$ provides the measure of what extent p is close to be true given the incomplete knowledge K and the similarity S. In particular, if the set of worlds is finite then, $I_S(p \mid K) = 1$ if, and only if, $K \models p$. On the other hand, the value of $C_S(p \mid K)$ provides the measure to what extent p can be considered compatible with the available knowledge K. In particular, in the finite case, $C_S(p \mid K) = 1$ if, and only if, $K \wedge p$ is consistent, or in other words $K \not\models \neg p$. Observe that, when the evidence K is *complete*, i.e. when K has only one model w_0, both measures coincide, $I_S(p \mid K) = C_S(p \mid K) = \sup\{S(w, w_0) \mid w \models p\}$.

It is easy to show that, given a fixed K, the measure $C_S(\cdot \mid K)$ is a possibility measure (Dubois *et al.* 1988) since it holds

1. $C_S(\top \mid K) = 1$
2. $C_S(\perp \mid K) = 0$
3. $C_S(p \vee q \mid K) = \max(C_S(p \mid K), C_S(q \mid K))$.

It is also very easy to check that the possibility distribution $\pi_{S,K} : \mathcal{W} \to [0, 1]$ inducing the measure $C_S(\cdot \mid K)$ is defined by

$$\pi_{S,K}(w) = \sup_{w' \models K} S(w, w').$$

Notice that $\pi_{S,K}(w) = 1$ for any world w satisfying K.

The implication and consistency measures for a given K are not dual in the usual De Morgan sense, in other words, the implication measure $I_S(\cdot \mid K)$ is not the dual necessity measure of $C_S(\cdot \mid K)$. Moreover, $I_S(\cdot \mid K)$ it is not even a necessity measure in the possibilistic sense, since in general

$$I_S(p \wedge q \mid K) \neq \min(I_S(p \mid K), I_S(q \mid K)).$$

Actually, the only remarkable decomposability property of I_S is the following one:

$$I_S(p \mid K_1 \vee K_2) = \min(I_S(p \mid K_1), I_S(p \mid K_2)).$$

Nevertheless, implication and consistency measures do behave in a dual way in some sense. Notice first that having a similarity S defined on the set of possible worlds allows us to define for each proposition p a system of nested sets of worlds around the set of models of p. Define for each $\alpha \in [0, 1]$, the set

$$[p]_\alpha = \{w \in \mathcal{W} \mid \exists w' : w' \models p,\ S(w, w') \geq \alpha\}.$$

That is, $w \in [p]_\alpha$ if w is similar to the degree α (at least) to some model of p. All the sets $[p]_\alpha$ include $[p]$, and thus they can be seen as approximations of $[p]$, of course, if $\alpha_1 \leq \alpha_2$ then $[p]_{\alpha_2} \subseteq [p]_{\alpha_1}$. Moreover, if S is separating then $[p]_1 = [p]$. Actually the set of approximations $\{[p]_\alpha \mid \alpha \in [0, 1]\}$ is nothing but a system of spheres, in the sense of Lewis, centered around $[p]$. Now, it is easy to check the following conditions:

1. $C_S(p \mid K) \geq \alpha$ iff $[K] \cap [p]_\alpha \neq \emptyset$
2. $I_S(p \mid K) \geq \alpha$ iff $[K] \subseteq [p]_\alpha$

[1] A t-norm is a binary operation in the real unit interval [0, 1] which is associative, commutative, non-decreasing, and having 1 as neutral element.

In classical logic, K entails p only if $[K] \subseteq [p]$ holds. Therefore, looking at the second condition, one immediately sees that implication measures lead in a very natural way to a notion of graded approximate entailment (Dubois *et al.* 1997):

$$K \models_S^\alpha p \text{ iff } I_S(p \mid K) \geq \alpha \quad (\text{ iff } [K] \subseteq [p]_\alpha \;)$$

Notice that from this perspective $I_S(p \mid K)$ provides the greatest α such that $K \models_S^\alpha p$ holds. For each α, $\models_S^\alpha$ is a monotonic entailment since if $K \models_S^\alpha p$ and $K' \models K$ then $[K'] \subseteq [K] \subseteq [p]_\alpha$. This type of approximate entailment for a fuzzy reflexive relation S has been characterized in (Dubois *et al.* 1997) by the following set of postulates:

1. *Nestedness*: If $p \models^\alpha q$, then $p \models^\beta q$ for $\beta \leq \alpha$.
2. *Boundary-1*: $p \models^1 q$ iff $p \models q$
3. *Boundary-0*: $p \models^0 q$ for every $p, q \in \mathcal{L}$.
4. *Left Or*: $p \vee q \models^\alpha r$ iff $p \models^\alpha r$ and $q \models^\alpha r$.
5. *Right Or*: if p is a maximal conjunction, then $p \models^\alpha q \vee r$ iff $p \models^\alpha q$ or $p \models^\alpha r$.
6. *Consistency Preservation*: if p is consistent, $p \models^\alpha \bot$ only when $\alpha = 0$.
7. *Continuity from below*: if $p \models^\beta q$ for each $\beta < \alpha$, then $p \models^\alpha q$.

Moreover, if S is symmetric then one has to add the postulate:

8. *Symmetry*: if p and q are maximal conjunctions, then $p \models^\alpha q$ iff $q \models^\alpha p$

whereas if S is $\otimes$-transitive then the postulate to be added is

9. $\otimes$*-Transitivity*: if $p \models^\alpha q$ and $q \models^\beta r$, then $p \models^{\alpha\otimes\beta} r$.

Non-monotonic inferences based on orderings

From an epistemological perspective it seems intuitively plausible that our expectations about the world do not have the same strength. In order to formalize this idea, Makinson and Gärdenfors in (Gärdenfors and Makinson 1994) assume that there is an ordering $\leq^E$ of the sentences in a given language $\mathcal{L}$. Thus, given two sentences p and q, $p \leq^E q$ should be interpreted as "q is at least as expected as p" or "p is at least as surprising as q" (we shall write "$p <^E q$" as an abbreviation for "not $q \leq^E p$"). Makinson and Gärdenfors propose three properties which, they argue, must be satisfied by any reasonable ordering. They are:

transitivity If $p \leq^E q$ and $q \leq^E r$, then $p \leq^E r$.

dominance If $p \models q$, then $p \leq^E q$.

conjunctiveness $p \leq^E (p \wedge q)$ or $q \leq^E (p \wedge q)$.

The authors point out that the first postulate on expectation ordering is very natural for an ordering relation, the second postulate says that a logically stronger sentence is always less expected and the third constraint concerns the relation between the degrees of expectation of a conjunction $p \wedge q$ and the corresponding degrees of p and q respectively. Note that the three conditions imply reflexivity ($p \leq^E p$) and connectivity (either $p \leq^E q$ or $q \leq^E p$). By way of comparison, we may mention that these axioms are three of the five conditions used in (Gärdenfors 1988) and (Gärdenfors and Makinson 1994) to define the notion of *epistemical entrenchment* for the logic of theory change.

Now we may introduce how these ordering can be used to determine when p nonmonotonically implies q by said that q follows from p together with all the propositions that are expected in the light of p. The natural idea, according to Makinson and Gärdenfors, is to require that the added sentences must be strictly more expected than $\neg p$ in the ordering. This motivates the following definition.

Definition 1 *A relation* $\mid\!\sim$ *between propositions is a* comparative expectation *entailment iff there is an ordering* $\leq^E$ *satisfying transitivity, dominance and conjuntiveness such that the following conditions holds:*

$$p \mid\!\sim q \quad \textit{iff} \quad p \models q \quad \textit{or there is a proposition } r \textit{ such that } p \wedge r \models q \textit{ and } \neg p <^E r$$

It has been proven by Makinson and Gärdenfors that comparative entailments satisfy the desirable properties of Supraclassicality, Left Logical Equivalence, And, Consistency Preservation, Cut, Or and Rational Monotony (see (Gärdenfors and Makinson 1994) for their definitions) and vice versa. So these properties characterize comparative entailments.

An alternative form to define an ordering $\leq^F$ between sentences is proposed by (Fariñas *et al.* 1994), it is called *possibility ordering* and it is required to satisfy the axioms of transitivity, dominance together with:

disjunctiveness $p \vee q \leq^F p$ or $p \vee q \leq^F q$

In this case, $p \leq^F q$ will denote that q is at least as possible as p.

As pointed out in (Fariñas *et al.* 1994), the dual of a possibility ordering, defined as $p \leq^F q$ iff $\neg q \leq^E \neg p$, is an *expectation* ordering in the sense of Gärdenfors and Makinson. So, if the condition $\neg p <^E r$ is changed by $\neg r <^F p$ in the definition 1, we shall obtain an equivalent comparative entailment. Furthermore, in (Fariñas *et al.* 1994) it is shown that the following three clauses are equivalent for a possibility ordering $\leq^F$:

1. There is a proposition r such that $p \wedge r \models q$ and $\neg r <^F p$.
2. $p \wedge \neg r <^F p$.
3. $p \wedge \neg r <^F p \wedge r$.

Last result allows to give different (but equivalent) versions of the definition 1.

Non-monotonic inferences induced by consistency measures

Since $C_S(\cdot \mid K)$ is a possibility measure, the ordering induced on formulas

$$p \leq_C q \text{ iff } C_S(p \mid K) \leq C_S(q \mid K),$$

read as q is at least as consistent (with K) as p, is a *qualitative possibility* relation in the sense of Dubois (Dubois 1986), i.e. the qualitative counterpart of a possibility measure. A qualitative possibility relation is a possibility ordering (as defined above) together with this further axiom

$Non-triviality: \perp <_C \top$

where $<_C$ is the strict part of the ordering $\leq_C$.

In (Fariñas *et al.* 1994), it is shown that qualitative possibility relations and possibility orderings generate the same family of nonmonotonic entailments.

If we remove in the above definition the first condition ($p \models q$), and since $C_S(\cdot \mid K)$ is a possibility measure, what we get is exactly the *possibilistic entailment* proposed by Benferhat et al. (Benferhat *et al.* 1997):

$$p \mid\!\sim_{pos} q \text{ iff } p \wedge q >_C p \wedge \neg q.$$

Then supraclassicality is lost when for instance $p = \perp$. Possibilistic entailment has been characterized by a slightly different set of seven postulates: Or, Restricted Reflexivity (RR), Strong Consistency Preservation (SCP), Left Logical Equivalence (LLE), Right Weakening (RW), And and Rational Monotony (RM). The representation theorem for the possibilistic entailment by Benferhat et al. (see e.g. (Benferhat *et al.* 1997)) says that for any consequence relation $\mid\!\sim$ satisfying these properties there is a possibility measure Π on formulas such that $p \mid\!\sim q$ iff $\Pi(p \wedge q) > \Pi(p \wedge \neg q)$. As a corollary we get the following.

Corollary 1 *A consequence relation $\mid\!\sim$ satisfies Or, RR, SCP, LLE, RW, And and RM iff there exists a proposition K and a similarity S on possible worlds such that $p \mid\!\sim q$ iff $C_S(p \wedge q \mid K) > C_S(p \wedge \neg q \mid K)$.*

The proof is just a consequence of the fact that for any possibility distribution π on the set of worlds $\mathcal{W}$ one can always find a subset $E \subseteq \mathcal{W}$ and a similarity function S such that $\pi(w) = \sup\{S(w, w') \mid w \in E\}$ (see Section 2) . Namely, we have just to take K such that $[K] = E$ and check that the consistency measure $C_S(\cdot \mid K)$ is just the possibility measure generated by the distribution π.

Non-monotonic inferences induced by implication measures

As we have already mentioned, an implication measure $I_S(\cdot \mid K)$ does not verify any interesting decomposability property and this makes quite difficult to grasp which properties may satisfy an ordering on propositions defined as

$$p \leq_I q \text{ iff } I_S(p \mid K) \leq I_S(q \mid K)$$

However, one can find a way out by reasoning contrapositively. Namely, if $I_S(p \mid K)$ measures to what extent p is implied by K, one can also consider another implication-like index $L_S(p \mid K)$ measuring to what extent $\neg p$ implies $\neg K$. Hence we define

$$L_S(p \mid K) = I_S(\neg K \mid \neg p).$$

It is easy to show now that, given a fixed consistent K, the measure $L_S(\cdot \mid K)$ fulfills the following properties:

1. $L_S(\top \mid K) = 1$
2. $L_S(p \wedge q \mid K) = \min(L_S(p \mid K), L_S(q \mid K))$

but fails to satisfy $L_S(\perp \mid K) = 0$. This means that $L_S(\cdot \mid K)$ is very near to a necessity measure. The failure of this last property acknowledges the fact that the distribution $\pi^*_{S,K} : \mathcal{W} \to [0, 1]$ generating $L_S(\cdot \mid K)$ as a necessity measure,

$$L_S(p \mid K) = \inf_{w \not\models p} 1 - \pi^*_K(w)$$

may be actually non-normalized. Indeed, one can check that $\pi^*_{S,K}$ is defined by

$$\pi^*_{S,K}(w) = 1 - \pi_{S,\neg K}(w) = 1 - \sup_{w' \not\models K} S(w, w')$$

and it is clear that nothing forces $\pi^*_{S,K}$ to reach the value 1 for some model. Moreover one has $\pi^*_{S,K}(w) = 0$ for every world w such that $w \not\models K$. Non-normalized possibility distributions play a main role to represent partially inconsistent information (Benferhat *et al.* 1997).

The ordering induced by L_S,

$$p \leq_L q \text{ iff } L_S(p \mid K) \leq L_S(q \mid K),$$

is indeed a genuine expectation ordering, that is, it satisfies transitivity, dominance and conjunctiveness.

Because of the lack of normalization of π^*_K, in general the ordering $\leq_L$ is not a qualitative necessity relation, the dual of a qualitative possibility relation, since the condition $\perp <_L \top$ may not hold if S is not separating. A formal study of this notion of necessity (where it is not required that $n(\perp) = 0$) is given in (Biacino and Gerla 1992).

The comparative nonmonotonic inference relation induced by $\leq_L$ is therefore defined as follows:

$$p \mid\!\sim_L q \text{ iff either } p \models q \text{ or } \; p \to q >_L p \to \neg q,$$

where $\to$ denotes material implication. Again, this nonmonotonic inference relation $\mid\!\sim_L$ satisfies the same properties as $\mid\!\sim_C$, that is, Supraclassicality (SC), Left Logical Equivalence (LLE), And, Or, Rational Monotonicity (RM), Consistency Preservation (CP) and Cummulativity (CU). Therefore we can also provide the following corollary.

Corollary 2 *An inference relation $\mid\!\sim$ satisfies SC, LLE, And, Or, RM, CP and CU iff there exists a proposition K and a similarity S on possible worlds such that $p \mid\!\sim q$ iff either $p \models q$ or $L_S(p \to q \mid K) > L_S(p \to \neg q \mid K)$.*

Notice that if S is separating, the non-triviality axiom for $\leq_L$ holds and an analogous to Corollary 1 would also hold for an inference relation defined as $p \mid\!\sim_{nec} q$ iff $L_S(p \to q \mid K) > L_S(p \to \neg q \mid K)$.

It is interesting to also express $\mid\!\sim_L$ in terms of the graded approximate entailment $\models^\alpha$ introduced in Section 2. Just by applying the definitions, it turns out that the following condition holds:

$$p \mid\!\sim_L q \text{ iff either } p \models q \text{ or}$$
$$\text{there exists } \alpha \in [0,1] \text{ such that } p \wedge \neg q \models^\alpha \neg K \text{ and}$$
$$p \wedge q \not\models^\alpha \neg K.$$

In other words, q nonmonotonically follows from p, in a context of K and S, if $\neg K$ is approximately entailed by $\neg(p \to q)$ to a higher degree than by $\neg(p \to \neg q)$, or roughly speaking, when falsifying $p \to q$ falsifies K *more* than when falsifying $p \to \neg q$.

Orderings and sphere systems

In this section we first recall Lewis' sphere systems and show two ordering relations on sentences that can be generated by these systems.

Definition 2 *An universal system of spheres* $\mathbf{SP}$ *centered on* K *is any collection of subsets of worlds satisfying:*

1. $\mathbf{SP} = \{S_i\}_{i \in I}$ *is a nested system of sets called spheres, i.e.* $\mathbf{SP}$ *is totally ordered by* $\subseteq$.
2. K *is the* $\subseteq$*-minimun of* $\mathbf{SP}$.
3. $\mathcal{W}$ *is the largest element of* $\mathbf{SP}$.

Thus any sphere system centered on K is of the form $\mathbf{SP} = \{K \ldots \subseteq S_i \subseteq \ldots \subseteq \mathcal{W}\}$. Now, we can give two definitions of orderings on sentences arising in a natural way from a given sphere system $\mathbf{SP}$ centered on K.

Definition 3 *An* intersection *ordering is defined by:* $p \leq_K^{\min} q$ *iff* $\forall S_i \in \mathbf{SP}$ *if* $[q] \cap S_i \neq \emptyset$ *then* $[p] \cap S_i \neq \emptyset$.

In (Niiniluoto 1987, Pag. 200), the author interprets last order as an indication that p is at least as near to the truth K as q. There, he also points out that the following order indicates that p is at least as informative about the truth K as q.

Definition 4 *An* inclusion *ordering is defined by:* $p \leq_K^{\max} q$ *iff* $\forall S_i \in \mathbf{SP}$ *if* $[q] \subseteq S_i$ *then* $[p] \subseteq S_i$.

Note that given a finite universal sphere system $\mathbf{SP} = \{K \subseteq S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n \subseteq W\}$, inclusion and intersection orderings are dual in the following sense. $p \leq_K^{\min} q$ in $\mathbf{SP}$ if and only if $p \leq_{\overline{K}}^{\max} q$ in $\overline{\mathbf{SP}}$, where $\overline{\mathbf{SP}} = \{W - S_n \subseteq W - S_{n-1} \subseteq W - S_{n-2} \subseteq \ldots \subseteq W - K \subseteq W\}$ and $\overline{K} = \{p \in \mathcal{L} \mid [p] \subseteq W - S_n\}$.

Proposition 1 *Given an universal sphere system* $\mathbf{SP}$ *centered in* K*, the order* $\leq_K^{\min}$ *as defined above satisfies the following properties:*

1. $\top \leq_K^{\min} p$.
2. *If* $p \models q$ *then* $q \leq_K^{\min} p$.
3. *If* $p \leq_K^{\min} q$ *and* $q \leq_K^{\min} r$ *then* $p \leq_K^{\min} r$.
4. $p <_K^{\min} \bot$.
5. $p \leq_K^{\min} p \vee q$ *or* $q \leq_K^{\min} p \vee q$.
6. $K \leq_K^{\min} p$.

Proposition 2 *Given an universal sphere system* $\mathbf{SP}$ *centered in* K*, the order* $\leq_K^{\max}$ *as defined above satisfies the following properties:*

1. $p \leq_K^{\max} \top$.
2. *If* $p \models q$ *then* $p \leq_K^{\max} q$.
3. *If* $p \leq_K^{\max} q$ *and* $q \leq_K^{\max} r$ *then* $p \leq_K^{\max} r$.
4. $\bot <_K^{\max} p$.
5. $p \vee q \leq_K^{\max} p$ *or* $p \vee q \leq_K^{\max} q$.
6. $p \models K$ *or* $K <_K^{\max} p$.

$\leq_K^{\max}$ as follows:

Theorem 1 *Expectation orderings and intersection orderings (resp. inclusion orderings) are interdefinable by considering:*

$$p \leq^E q \Leftrightarrow \exists K : \neg p \leq_K^{\min} \neg q \;\; (resp.\; \exists K : \neg q \leq_K^{\max} \neg p)$$

Proof: From right to left, the proof is easy and we omit it. For the opposite direction, we rewrite a definition given in (Gärdenfors and Makinson 1994). There they define $p \leq_K^{\min} q$ if and only if both $\neg p \leq^E \neg q$ and it is not the case that $\neg p \equiv \top$ while $\neg q \not\equiv \top$ where $K = \{p \in \mathcal{L} \mid p \leq_K^{\min} \neg p\}$ (resp. $p \leq_K^{\max} q$ if and only if both $\neg q \leq^E \neg p$ and it is not the case that $\neg q \equiv \top$ while $\neg p \not\equiv \top$ with $K = \{p \in \mathcal{L} \mid \neg p \leq_K^{\min} p\}$). We omit the verification, which is tedious but not difficult.

Finally we can show that our consistency and implication-based orderings introduced in Sections 4 and 5 are indeed intersection and inclusion orderings in a suitable sphere system.

Theorem 2 *Let* K *be a consistent proposition and* $S : \mathcal{W} \times \mathcal{W} \to [0,1]$ *a similarity relation on possible worlds. Define the sphere systems centered on* K *by* $\mathbf{SP}_{K,S} = \{[K]_\alpha \mid \alpha \in [0,1]\}$. *Then it holds that:*

1. $C(p \mid K) \leq C(q \mid K)$ *iff* $q \leq_K^{min} p$ *in* $\mathbf{SP}_{K,S}$.
2. $L(p \mid K) \leq L(q \mid K)$ *iff* $\neg q \leq_{\neg K}^{max} \neg p$ *in* $\mathbf{SP}_{\neg K,S}$.

Therefore, sphere systems provide a wide enough semantical framework to encode our similarity-based orderings and nonmonotonic inference relations. Sphere systems were already linked to graded similarity structures in (Rodriguez *et al.* 1996; Dubois *et al.* 1997).

Related and future work

The notion of similarity (or distance) between situations plays a central role in some kinds of reasoning like approximate, case-based or interpolative. While the logical formalisms underlying them are mainly monotonic (see for instance (Dubois and Prade 1994; Ying 1994; Dubois *et al.* 1997; Plaza *et al.* 1998)), sometimes it is necessary to have nonmonotonic entailments based on similarity like Lehmann's Stereotypical reasoning (Lehmann 1998), or a very recent proposal to provide a logical interpretation (in terms of nonmonotonic inferences) of dilation and erosion operators used in mathematical morphology techniques (Bloch and Lang 2000). With this idea in mind, we have focused in this paper some ways of nonmonotonic inference based in similarity measures between situations, in particular, those that can be interpreted in terms of consistency and implication measures. As already mentioned in the Introduction,these measures deal with a notion different from uncertainty or vagueness. They provide a *truth-like* value of a sentence as its degree of "proximity to the truth", even though it may false or unlikely. This degree accounts for the "distance" that separates the models of this sentence and the situations compatible with the available information (what is known from the "reality"). In this sense, we are proposing non-monotonic entailments based on the degree of truth-likeness of sentences.

We should remark that, fixed a non contradictory context K and a separating similarity function S, the non-monotonic operators generated by these measures are never opposed one to another, i.e. it is never the case that $p \mid\!\sim_C q$ and $p \mid\!\sim_L \neg q$. Indeed, the following analysis by cases shows it:
Let p q and K be non contradictory propositions.

Case: $K \models p$, $K \not\models q$ **and** $K \not\models \neg q$
Then $C_S(p \wedge q \mid K) = C_S(p \wedge \neg q \mid K) = 1$.

Case: $K \models p$ **and** $K \models q$
Then $C_S(p \wedge q \mid K) > C_S(p \wedge \neg q \mid K)$ and $L_S(p \to q \mid K) > L_S(p \to \neg q \mid K)$.

Case: $K \models p$ **and** $K \models \neg q$
Then $C_S(p \wedge q \mid K) < C_S(p \wedge \neg q \mid K)$ and $L_S(p \to q \mid K) < L_S(p \to \neg q \mid K)$.

Case: $K \models \neg p$
Then $L_S(p \to q \mid K) = L_S(p \to \neg q \mid K) = 1$.

Hence, it would be possible to consider a new operator by combining them as follows: $p \mid\!\sim_{C,L} q$ iff $p \mid\!\sim_C q$ and/or $p \mid\!\sim_L q$. The properties of such new class of non-monotonic operators would deserve investigation in the near future.

On the other hand, one way to extend the approach would be to consider graded versions of nonmonotonic entailments. One possibility is the following as follows: given a non-monotonic inference relation $\mid\!\sim$, and a fuzzy similarity relation S, a graded nonmonotonic entailment $\mid\!\sim^\alpha$ defined by:

$$p \mid\!\sim^\alpha q \text{ iff } p \mid\!\sim q_\alpha$$

where q_α stands for a proposition whose set of models is $[q]_\alpha = \{w \mid S(w, w') \geq \alpha \text{ for some } w' \models q\}$. If $\mid\!\sim$ is a comparative inference relation defined by a possibility ordering $\leq$, then it comes down to write

$$p \mid\!\sim^\alpha q \text{ iff } p \models^\alpha_S q \text{ or } p \wedge q_\alpha > p \wedge \neg q_\alpha.$$

If S is separating, we recover the original inference relation $\mid\!\sim$ when $\alpha = 1$. Of course other alternatives are also possible and further research is needed to explore them.

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References

S. Benferhat, D. Dubois, and H. Prade. Nonmonotonic reasoning, conditional objects and possibility theory. *Artificial Intelligence*, 92:259–276, 1997.

L. Biacino and G. Gerla. Generated necessities and possibilities. *Int. J. of Intell. Systems*, 7:445–454, 1992.

I. Bloch and J. Lang. Towards morphologics. *Submitted*, 2000.

D. Dubois and H. Prade. Similarity-based approximate reasoning. In J.M. Zurada, R.J. Marks II, and X.C.J. Robinson, editors, *Computational Inelligence Imitating Life*, pages 69–80, Orlando, 1994. IEEE Press.

D. Dubois and H. Prade. Comparison of two fuzzy set-based logics: similarity logic and posibilistic logic. In *FUZZ-IEEE'95*, pages 1319–1326, Yokohama, Japan, July 1995. IEEE Press.

D. Dubois, J. Lang, and H. Prade. Possibilistic logic. In D.M. Gabbay, C.J. Hogger, J.A. Robinson, and D.Nute, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming*, volume 3, pages 439–514. Oxford University Press, 1988.

D. Dubois, F. Esteva, P. Garcia, L. Godo, and H. Prade. A logical approach to interpolation based on similarity relations. *Int. J. of Approximate Reasoning*, 17(1):1–36, 1997.

D. Dubois. Belief structures, possibility theory, decomposable confidence measures on finite sets. *Comput. Artif. Intelligence*, 5(5):403–417, 1986.

F. Esteva, P. Garcia, L. Godo, and R. Rodriguez. A modal account of similarity-based reasoning. *Int.J.of Approximate Reasoning*, 16(3/4):235–261, 1997.

L. Fariñas del Cerro, A. Herzig, and J. Lang. From ordering-based nonmonotonic reasoning to conditional logics. *Artificial Intelligence*, 66:375–394, 1994.

P. Gärdenfors and M. Makinson. Nonmonotonic inference based on expections. *Artificial Intelligence*, 65:197–245, 1994.

P. Gärdenfors. *Knowledge in Flux: Modeling the Dynamics of Epistemic States.* MIT Press, Cambridge, 1988.

D. Lehmann. Stereotypical reasoning: logical properties. *Logic Journal of the IGPL*, 6(1):49–58, 1998.

D. Lewis. *Contrafactuals.* Oxford Basil Blackwell, London, 1973.

I. Niiniluoto. *Truthlikeness*, volume 185. Syntheses Library, Reidel, Holland, 1987.

E. Plaza, F. Esteva, P. Garcia, L. Godo, and R. Lopez de Mantaras. A logical approach to case-based reasoning using fuzzy similarity relations. *Journal of Information Sciences*, 106:105–122, 1998.

R: Rodriguez, P. Garcia, and L. Godo. Using fuzzy similarity relations to revise and update a knowledge base. *Mathware and Soft Computing*, 3(3):357–370, 1996.

E. Ruspini. On the semantics of fuzzy logic. *Int. J. of Approximate Reasoning*, 5:45–88, 1991.

T. Williamson. First-order logics for comparative similarity. *Notre Dame Journal of Formal Logic*, 29(4):457–481, 1988.

M. Ying. A logic for approximate reasoning. *Journal of Symbolic Logic*, 59(3):830–837, 1994.

Appendix

We recall here the above mentioned properties of comparative entailments.

Supraclassicality
If $p \models q$ then $p \mid\!\sim q$.

Left Logical Equivalence
If $\models p \leftrightarrow q$ and $p \mid\!\sim r$ then $q \mid\!\sim r$

Consistency Preservation
If $p \mid\!\sim \bot$ then $p \models \bot$.

Right Weakening
If $\models p \rightarrow q$ and $q \mid\!\sim r$ then $p \mid\!\sim r$.

And
If $p \mid\!\sim q$ and $p \mid\!\sim r$ then $p \mid\!\sim q \wedge r$.

Weak Conditionalization
If $p \mid\!\sim q$ then $\models p \rightarrow q$.

Cumulativity
If $p \mid\!\sim q$ and $q \models p$ then $p \mid\!\sim r$ iff $q \mid\!\sim r$.

Or
If $p \mid\!\sim q$ and $r \mid\!\sim q$ then $p \vee r \mid\!\sim q$

Rational Monotony
If $p \not\mid\!\sim \neg q$ and $p \mid\!\sim r$ then $p \wedge q \mid\!\sim r$.

Weak Rational Monotony
If $\not\mid\!\sim \neg p$ and $\mid\!\sim p \rightarrow q$ then $p \mid\!\sim q$

Cautious Monotony
If $p \mid\!\sim q$ and $p \mid\!\sim r$ then $p \wedge q \mid\!\sim r$.

Cut
If $p \mid\!\sim q$ and $p \wedge q \mid\!\sim r$ then $p \mid\!\sim r$.