A Statistical Probability Theory for a Symbolic Management of Quantified Assertions

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Abstract

In this paper we present a new approach to a symbolic treatment of quantified statements having the following form "Q A's are B's", knowing that A and B are labels denoting sets, and Q is a linguistic quantifier interpreted as a proportion evaluated in a qualitative way. Our model can be viewed as a symbolic generalization of statistical conditional probability notions as well as a symbolic generalization of the classical probabilistic operators. Our approach is founded on a symbolic finite M-valued logic in which the graduation scale of M symbolic quantifiers is translated in terms of truth degrees. Moreover, we propose symbolic inference rules allowing us to manage quantified statements.

1 Introduction

In the natural language, one often uses statements qualifying statistical information like "Most students are single". Usually they are represented more formally under the form "Q A's are B's" where A and B are labels denoting individuals sets and Q is a linguistic quantifier. Zadeh (Zadeh 85) distinguishes between two types of *quantifiers: absolute* and *proportional*. An absolute quantifier evaluates the number of individuals of B in A. While a proportional quantifier evaluates the proportion of individuals of B in A. The proportional quantifiers can be precise or vague. A *precise quantifier* translates an interval of proportions having precise bounds exemplified by "10%", "Between 10 and 20%", etc. While a *vague quantifier* translates an interval proportions having fuzzy bounds. Thus the vague proportional quantifiers express qualitatively proportions. A *proportional quantifier* can be viewed as a kind of probabilities assigned to classes of individuals. So, several approaches based on the theory of probabilities have been proposed ((Kyburg 83), (Pollock 90), (Dubois & Prade 88), (Bacchus 90), (Akdag et al. 92), (Bacchus et al. 97), (Jaeger 95)) for the modeling of precise proportional quantifiers. Other probabilistic approaches, such as those proposed by ((Nilsson 86), (Paass 88), (Cheeseman 88)), do not enable an adequate representation of proportional quantifiers, since these approaches are generally introduced to treat uncertainty. These authors interpret the probability degrees assigned to propositions as degrees of certainty or beliefs in the truth of these propositions. They represent statistical assertions of type "Q A's are B's" as uncertain rules of the form: "if A then B" with a belief subjective degree in the truth of the rule (A and B are interpreted as propositions). It has been pointed out by Bacchus (Bacchus 90) that a confusion in the representation is made between the probabilities interpreted as certainty degrees assigned to propositions about particular individuals and those interpreted as proportions assigned to classes of individuals. The *probabilities* of the first type are called *subjective* and the second *statistical.* The statistical probability that corresponds to the proportion is a particular case of probabilities where the distribution is uniform over the finite reference set. For example, the statistical probability attached to a subset A of the finite reference set Ω, Prop(A), is equal to the absolute proportion of individuals of A, i.e., Prop(A)=$|A|/|\Omega|$. Similarly, if A and B are two subsets of Ω, the relative proportion of individuals of B in A is expressed by the conditional statistical probability, Prop(B|A), with Prop(B|A)=Prop(A∩B)/Prop(A)=$|A\cap B|/|A|$. Some probabilistic approaches ((Adams 75), (Pearl 91), (Bacchus 90), (Bacchus et al. 97)) are interested in a qualitative modeling of the proportional quantifier "Most" or "Almost-all" in the context of default reasoning. The approaches based on the fuzzy set theory ((Zadeh 85), (Yager 85), (Yager 86), (Dubois & Prade 88)) deal with a vague proportional quantifier as a fuzzy number of the interval [0,1] which can be manipulated by using the fuzzy arithmetic. For example, the membership function of "Most" evaluates the degree to which a given proportion r is compatible with the quantifier "Most". The representation of quantified statements involving fuzzy sets is based on the concept of fuzzy subset cardinality. Recently, Dubois *et* al. (Dubois et al. 92) have proposed a semi-numerical approach to the vague quantifiers based upon the numerical results obtained in ((Dubois & Prade 88), (Akdag et al. 92)) for precise quantifiers. It is concerned with a suitable ordered partition of the unit interval [0,1] in several subintervals covering [0,1], subinterval representing a vague quantifier. The subintervals obtained by applying the inference rules (on the precise quantifiers) to subintervals representing to the vague quantifiers are approximately associated to subintervals of vague quantifiers.

In this paper, we propose a purely *symbolic approach to represent vague proportional* quantifiers with a statistical interpretation in terms of proportions evaluated in qualitative way[1]. More precisely, we define a semantic model of statistical probabilities representation inspired by Bacchus's model (Bacchus 90). The semantic model is built on the basis of the finite M-valued predicates symbolic logic introduced by Pacholczyk (Pacholczyk 92) for a symbolic treatment of vague information. We have introduced a new predicate in the language of this logic that takes into account the notion of proportions. A graded scale symbolic quantifiers is associated with graded scale of truth symbolic degrees of this predicate. In Section 2, we briefly present the many-valued symbolic logic. Section 3, describes our symbolic representation of statistical probability. The Axioms governing this representation are presented in Section 4. Section 5 deals with certain properties generalizing symbolically some classical properties. Inference rules manipulating quantified statement are presented in Section 6.

2 M-valued predicates logic

As noted before, the semantic model of statistical probabilities proposed here is built on the substrate of the M-valued predicate logic proposed by Pacholczyk (Pacholczyk 92). Instead of translatting statistical probabilities in terms of probabilistic equalities, it is more convenient to consider that statistical probability is representable by a multiset $\mathfrak{S}$ and then that a statistical probability stands for the degree to which this multiset is satisfied. That is why we refer to an M-valued logic. More formally, one is led to enrich the M-valued logic by adjoining a particular M-valued predicate, denoted as **Prop** (Section 3), and by putting the axioms governing statistical probabilities. Let us briefly recall the notions of interpretation and satisfaction in this M-valued logic.

2.1 Algebraic Structures

Let M ≥ 2[2] be an odd integer. Let $\mathfrak{M}$ be the interval [1, M] totally ordered by the relation $\leq$, and n be the mapping defined by n(α) = M+1-α. Then, $\{\mathfrak{M}, \vee, \wedge, n\}$ is a De Morgan lattice with :$\alpha \vee \beta = \max(\alpha, \beta)$ and $\alpha \wedge \beta = \min(\alpha, \beta)$. Let $\mathfrak{L}_{\mathfrak{M}} = \{\tau_\alpha, \alpha \in \mathfrak{M}\}$ be a set of M elements totally ordered by the relation $\leq$such that : $\tau_\alpha \leq \tau_\beta \Longleftrightarrow \alpha \leq \beta$. Thus $\{\mathfrak{L}_{\mathfrak{M}}, \leq\}$ is a chain in which the least element is τ_1 and the greatest element is τ_M. We define in $\mathfrak{L}_{\mathfrak{M}}$ two operators and and a decreasing involution as follows : $\tau_\alpha \vee \tau_\beta = \tau_{max(\alpha,\, \beta)}$, $\tau_\alpha \wedge \tau_\beta = \tau_{\min(\alpha,\, \beta)}$ and $\backsim \tau_\alpha = \tau_{n(\alpha)}$. We can interpret $\mathfrak{L}_{\mathfrak{M}}$ as a set of linguistic truth degrees allowing to deal with vague predicates. For example, by choosing M = 7, we can introduce: $\mathfrak{L}_7 =$ {not-at-all-true, very-little-true, little-true, moderately-true, very-true, almost-true, totally-true}[3]. In a statement having the following form "x is v_αA", the term v_α linguistically expresses the degree to which the object x satisfies the concept A ((Pacholczyk 92), (Pacholczyk 94)). Each linguistic term v_α is associated to the truth degree τ_α-true. So, we have : "John is very tall" is true $\Longleftrightarrow$"John is tall" is very-true . In the following, the lattice $\{\mathfrak{L}_{\mathfrak{M}}, \vee, \wedge, \backsim\}$ will be used as the support of the representation of M truth degrees.

2.2 Interpretation and satisfaction of formulas

The many-valued predicates logic used here can be found in (Pacholczyk 92). Let $\mathfrak{L}$ be the many-valued predicates language and $\mathfrak{F}$ the set of formulas of $\mathfrak{L}$. We call an interpretation structure $\mathfrak{A}$ of $\mathfrak{L}$, the pair $< \mathfrak{D}, \{R_n \mid n \in N\} >$, where $\mathfrak{D}$ designates the domain of $\mathfrak{A}$ and R_n designates the multiset[4] associated with the predicate P_n of the language. We call a valuation of variables of $\mathfrak{L}$, a sequence denoted as v = $<v_0,...,v_{i-1},v_i,\ v_{i+1},...>$. The valuation v(i/a) is defined by v(i/a) = $<v_0,...,v_{i-1},a,v_{i+1},...>$.

Definition 1 *For any formula Φ of $\mathfrak{F}$, the relation of partial satisfaction "v satisfies Φ to a degree τ_α in-$\mathfrak{A}$" or "v τ_α-satisfies Φ in-$\mathfrak{A}$", denoted as $\mathfrak{A} \models_\alpha^v \Phi$, is defined recursively as follows :*
- $\mathfrak{A} \models_\alpha^v P_n(z_{i_1},...,z_{i_k}) \Longleftrightarrow < v_{i_1},...,v_{i_k} > \in_\alpha R_n$.
- $\mathfrak{A} \models_\alpha^v \neg\phi \Longleftrightarrow \mathfrak{A} \models_\beta^v \phi$ *with* $\tau_\alpha = \tilde{}\tau_\beta$,
- $\mathfrak{A} \models_\alpha^v \phi \cap \psi \Longleftrightarrow \{\mathfrak{A} \models_\beta^v \phi$ *and* $\mathfrak{A} \models_\gamma^v \psi$ *with* $\tau_\alpha = \tau_\beta \wedge \tau_\gamma\}$,
- .$\mathfrak{A} \models_\alpha^v \phi \cup \psi \Longleftrightarrow \{\mathfrak{A} \models_\beta^v \phi$ *and* $\mathfrak{A} \models_\gamma^v \psi$ *with* $\tau_\alpha = \tau_\beta \vee \tau_\gamma\}$,
- $\mathfrak{A} \models_\alpha^v \exists z_n \psi \Longleftrightarrow \tau_\alpha = Max\{\tau_\gamma \ /\mathfrak{A} \models_\gamma^{v(n/a)} \psi,\ a \in \mathfrak{D}\}$,
- $\mathfrak{A} \models_\alpha^v \forall z_n \psi \Longleftrightarrow \tau_\alpha = Min\{\tau_\gamma \ /\mathfrak{A} \models_\gamma^{v(n/a)} \psi,\ a \in \mathfrak{D}\}$.

Definition 2 *A formula Φ is said to be τ_α-true-in-$\mathfrak{A}$, if and only if, there exists a valuation v such that v τ_α-satisfies Φ in-$\mathfrak{A}$.*

3 Symbolic representation of the statistical probabilities

The representation of statistical probabilities requires the reference to sets of individuals and also to assign probabilities to these sets. To solve the first problem, we use the concept of *placeholder variables* in lambda abstraction used by Bacchus (Bacchus 90), where one considers that a Boolean open formula can refer to the

[1]This model extends previous works proposed in ((Khayata 98), (Khayata & Pacholczyk 98)).

[2]In the assertion x is $v_\alpha A$, the linguistic expression v_α expresses the degree to which x satisfies A. In order to include the translation of x is A, we have introduced the empty word $\emptyset$. So: x is A $\Leftrightarrow$ x is $\emptyset$A. This empty word $\emptyset$ will be the same in the assertion x is not A: x is not A $\Leftrightarrow$x is $\emptyset$ not A. So, the corresponding truth degree v_a will be such that $\sim(v_a) = v_a$. This propery implies that M should be an odd integer.

[3]Note that "not-at-all-true" and "totally-true" correspond respectively with the classical truth values "false" and "true".

[4]The multiset theory is an axiomatic approach to the fuzzy set theory. In this theory, x $\in_\alpha$A, the membership degree to which x belongs to A, corresponds with μ_A(x) =α in the fuzzy set theory of Zadeh (Zadeh 65).

set of all instances of its free variables specified as placeholders, satisfying the formula. So given a many-valued predicates language $\mathfrak{L}$, for an interpretation $\mathfrak{A}$* with domain of discourse Ω and $\mathfrak{C}$ the set of open well-formed formulas without bound variables ϕ of $\mathfrak{F}$ such that, for any valuation v of Ω, ϕ totally satisfied in-$\mathfrak{A}$* or not at all satisfied in-$\mathfrak{A}$. *So*: $\mathfrak{C}= \{\phi \in \mathfrak{F} \mid \forall v,\ \mathfrak{A}^*\models_M^v \phi$ or $\mathfrak{A}^*\models_1^v \phi\}$. Since formulas of $\mathfrak{C}$ contain only free variables, we can consider that free variables of formulas of $\mathfrak{C}$ stand implicitly for placeholder variables. Thus in interpretation $\mathfrak{A}$*, each formula of $\mathfrak{C}$ will be able to make reference to the subset of individuals of Ω that satisfy this formula.

In order to define the symbolic statistical probabilities assigned to subsets referred by formulas of $\mathfrak{C}$, we add to the language $\mathfrak{L}$, a new many-valued unary predicate, denoted as **Prop**, defined over formulas of $\mathfrak{C}$ which qualitatively takes into account the notion of proportions of sets referred by formulas of $\mathfrak{C}$. We are going to extend the structure interpretation of the language $\mathfrak{A}$* to $\mathfrak{A}$ with domain $\Omega \cup \mathfrak{C}$, and we suppose that the variable φ designates the argument of **Prop**, and that any valuation v comprises v_0 that is associated to φ .

Definition 3 *The predicate* **Prop** *is defined as follows* [5]*:*

- For any interpretation $\mathfrak{A}$, $\forall\varphi \in \mathfrak{C}$, **Prop**$(\varphi)\in \mathfrak{F}$.

- Any interpretation $\mathfrak{A}$ associates to **Prop** *a multiset of $\mathfrak{C}$, denoted as $\mathfrak{S}$, so that for all valuation v, if φ is an element of $\mathfrak{C}$, we have: $\mathfrak{A} \models_\alpha^{\mathfrak{v}(o/\phi)}$* **Prop**$(\varphi) \iff <\phi>\in_\alpha \mathfrak{S} \iff$ **Prop**(ϕ) *is τ_α-true-in-$\mathfrak{A}$. If no confusion is possible $\mathfrak{A} \models_\alpha$***Prop**$(\phi)$ *stands for $\mathfrak{A} \models_\alpha^{\mathfrak{v}(o/\phi)}$***Prop**$(\varphi)$.

In (Pacholczyk 94) the main objective was the definition of an uncertainty concept within a symbolic theory of probabilities. Here, it concerns the representation of symbolic quantification. So, the semantic associated to **Prop** is different from the one chosen for **Prob** in (Pacholczyk 94).

Definition 4 *Let $\mathfrak{Q}_\mathfrak{M}$ be the set of the vague proportional quantifiers : $\mathfrak{Q}_\mathfrak{M}=\{Q_\alpha,\ \alpha \in[1,M]\}$. Then, "$\mathfrak{A} \models_\alpha Prop(\phi)$" will mean that "$Q_\alpha$ individuals of Ω totally satisfy ϕ in-$\mathfrak{A}$", if and only if, the subset referred by ϕ belongs to the multiset $\mathfrak{S}$ with a degree τ_α .*

Thus Q_α is considered as the symbolic degree of statistical probability of the set referred by ϕ in $\mathfrak{A}$. Therefore, each symbolic truth degree τ_α-true of $\mathfrak{L}_\mathfrak{M}$ of Prop(ϕ), is associated to a symbolic degree of statistical probability (proportion) of the set referred by ϕ, i.e., to a linguistic quantifier Q_α of $\mathfrak{Q}_\mathfrak{M}$.

Example 1 *By choosing $M = 7$, we can introduce : $\mathfrak{Q}_7$ = {none, very-few (or almost-none), few, about half, most, almost-all, all} that corresponds to the symbolic degrees of statistic probability.*

The idea proposed in Pacholczyk (Pacholczyk 94) for the representation of "Q A's are B's" was to interpret it as a symbolic conditional uncertainty of the event B given A. Our interpretation will be in terms of the symbolic relative (or conditional) proportion of individuals of B in A. Therefore, as in (Pacholczyk 94), we can generalize the classical definition of conditional statistical probability in a symbolic context, by using a "symbolic probabilistic division" operator, denoted as **C**, or equivalently a "symbolic probabilistic multiplication" operator, denoted as **I**. These two operators have been defined in (Pacholczyk 94) (see also (Xiang et al. 90)). The operator **I** is an application of $\mathfrak{Q}_\mathfrak{M}^2$ into $\mathfrak{Q}_\mathfrak{M}$, that verifies the classical properties of the probabilistic multiplication (commutativity, absorbent element: Q_1, identity: Q_M, monotony, associativity, idempotence: Q_2). The operator **C** is an application of $\mathfrak{Q}_\mathfrak{M}^2$ into $\mathfrak{P}(\mathfrak{Q}_\mathfrak{M})$. **C** is deduced from **I** by a unique way as follows: $Q_\mu \in$**C**$(Q_\alpha,\ Q_\lambda)\Longleftrightarrow Q_\lambda=$**I**$(Q\alpha,\ Q\mu)$. Among the different tables of the operator **C** which verify the axioms chosen in (Pacholczyk 94), in $\mathfrak{L}_7$ we have chosen table 1 presented in Annex A. The corresponding operator **I** is defined in Annex A by Table 2.[6]

Definition 5 *Let ϕ and ψ be formulas of $\mathfrak{C}$, we are going to introduce the symbolic conditional statistical probability, "$\mathfrak{A} \models_\mu Prop(\psi|\phi)$", expressing the symbolic degree of the relative proportion of individuals that satisfy the formula ψ among those satisfying φ. It will be defined by the symbolic division of the symbolic degree of $Prop(\psi \wedge \phi)$ by that of $Prop(\phi)$ as follows: $\{\mathfrak{A} \models_\alpha Prop(\phi),\ \mathfrak{A} \models_\lambda Prop(\psi \wedge \phi)\} \Longrightarrow \mathfrak{A} \models_\mu Prop(\psi|\phi)$ with $Q_\mu \in C(Q_\alpha,\ Q_\lambda)$. "$\mathfrak{A} \models_\mu Prop(\psi|\phi)$" means that " Among the individuals of Ω which totally satisfy ϕ in-$\mathfrak{A}$, Q_μ totally satisfy ψ in-$\mathfrak{A}$"*

In order to obtain an equivalent manipulation of sets and formulas of type "Prop($\psi|\phi$)", we can use the notation of the partial inclusion of a set in another, proposed in (Akdag et al. 92). This can be done by considering that the symbolic degree of partial inclusion of a set in another, coincides with the symbolic degree of statistical probability of individuals of the second in the first.

Suppose that ϕ and ψ refer respectively to subsets A and B of Ω in the interpretation $\mathfrak{A}$. The equivalence between the two notations whose sense remains identical, allows us to use : "$A \subset_\mu B$" instead of "$\mathfrak{A} \models_\mu$Prop$(\psi|\phi)$".

It can be established as follows: $\mathfrak{A} \models_\mu$Prop$(\psi|\phi)$ $\Longleftrightarrow Q_\mu$ individuals of A, are individuals of B $\Longleftrightarrow A\subset_\mu B$. Then, we have : $A\subset_\mu B \Longleftrightarrow Q_\mu$ A's are B's. Then, knowing that $\top$ is a tautology, we obtain: $\mathfrak{A} \models_\alpha$Prop$(\phi) \Longleftrightarrow \mathfrak{A} \models_\alpha$Prop$(\phi|\top)$. So, we can use

[5] **Prop** has been introduced in a similar way as the predicate **Prob** in (Pacholczyk 94).

[6] It can be noted that the operators $\vee$, $\wedge$ and $\sim$ have been defined in the M-valued logic in order to deal with partial truth degrees. But, they do not allow us to govern the particular predicate **Prop**, since statistical probabilities are not truth-functional, in this sense that the statistical probability of a compound formula is not a function of the statistical probabilities of its parts. In order to propose a symbolic representation of the statistical probability, we have to define the operators governing this symbolic concept.

"$\Omega \subset_\alpha$A" instead of "$\mathfrak{A} \models_\alpha$Prop($\phi$)". Thus Definition 5 can equivalently be written with notation of sets as follows:

Definition 6 *Given an interpretation $\mathfrak{A}$, let us suppose that the formulas ϕ and ψ refer respectively to the subsets A and B of the universe Ω. Then, in terms of partial inclusion in the multiset theory, $\mathfrak{A} \models_\mu Prop(\psi|\phi)$ receives as translation $A \subset_\mu B$. In other words, "Among the individuals of Ω which totally satisfy ϕ in-$\mathfrak{A}$, Q_μ totally satisfy ψ in-$\mathfrak{A}$"* **is equivalent to say that** *"A* **is included into** *B* **with a degree** *Q_μ".*

Remark 1 *Previous definition gives us: if $\{\Omega \subset_\alpha A$, $\Omega \subset_\lambda A \cap B\}$ then $A \subset_\mu B$ with $Q_\mu \in C(Q_\alpha, Q_\lambda)$. This can be viewed as a symbolic generalisation of the classical property: Prop(B|A) = Prop(A∩B) / Prop(A) = |A∩B| / |A|.*

Let A and B be subsets of Ω . It is easy to prove the following properties:

Proposition 1 $A \subset_M B \Longleftrightarrow A \subset B \Longleftrightarrow A \cap B = A$.

Since the degree Q_M corresponds with the quantifier "All", then $\subset_M$ coincide with the inclusion in set theory.

Proposition 2 $A \subset_1 B \Longleftrightarrow A \cap B = \emptyset$.

Since the degree Q_1 corresponds with the quantifier "None", this means that A and B are disjoint.

Example 2 *By using $\mathfrak{L}_7$, let us suppose that the domain of discourse consists residents of the city V. Knowing that: "Most residents of the city V are young" and "Half of residents of the city V are young single". These assertions are respectively translated in our model by: $\Omega \subset_5$ Young and $\Omega \subset_4$ Young single. Definition 6 gives us: Young $\subset_\mu$Single with $Q_\mu \in C(Q_5, Q_4) = \{Q_5\}$. Then we obtain: "Most young people are single".*

4 Axiomatic of the symbolic statistical probabilities

We can now put the axioms governing the concept of symbolic statistical probabilities. Let A and B be subsets of Ω . The axioms are defined as follows:

Axiom 1 *$A \cap B \neq A$, $\Omega \subset_\alpha A$, $\Omega \subset_\alpha A \cap B$ and $Q_\alpha \in [\mathbf{Q_3}, \mathbf{Q_{M-1}}] \Longrightarrow A \subset_{\mathbf{M-1}} B$. (Axiom defining "Almost-all")*

Qualitatively the subsets A and A∩ B can have the same symbolic degree of proportions without being equal. This is the case, when A∩ B is equal to the set A without one or some individuals. This can qualitatively translate by saying that "A and A∩ B are almost equal" or "Almost all A's are B's". This is not always the case, when the proportion of A is very weak (associated with Q_2 = Very-few).

Axiom 2 *$\Omega \subset_\alpha A$, $Q_\alpha \in [\mathbf{Q_2}, \mathbf{Q_{M-1}}]$ and $A \subset_{\mathbf{M-1}} B \Longrightarrow \Omega \subset_\alpha A \cap B$. (Axiom defining "Almost-all")*

When we have "Almost all A's are B's", we know that A $\neq$ A∩B (*Cf.* Proposition 1), but we can say that A and A∩B are almost equal and therefor A and A∩ B have the same symbolic degree of proportions.

Axiom 3 *$\Omega \subset_\alpha A \Longleftrightarrow \Omega \subset_{n(\alpha)} \overline{A}$ with $n(\alpha) = M + 1 - \alpha$. (Axiom defining the dual quantifier)*

Generally the dual quantifier of Q_α corresponds with $Q_{n(\alpha)}$ ("Few" is the dual quantifier of "Most").

Axiom 4 *$\Omega \subset_\alpha A$, $\Omega \subset_\beta B$, $A \cup B \neq \Omega$ and $A \cap B = \emptyset \Longrightarrow \Omega \subset_r A \cup B$ with $Q_r \in S(Q_\alpha, Q_\beta)$. (Axiom defining the symbolic proportion of disjoint sets union)*

Classically, when A and B are disjoint, the absolute proportion of their union is the sum of their absolute. We put that if the union A and B is different from Ω (otherwise, the symbolic proportion degree of their union is evidently Q_M) and that they are disjoint, then the symbolic proportion degree of their union belongs to the "*symbolic sum*" of their symbolic proportion degrees. The symbolic sum denoted **S** is introduced in a way that it gives an interval containing one or two values. The lower bound of this interval is greater than or equal to each symbolic value of two arguments of S. Since the set A∪B is different from Ω, the maximal degree that can take the upper bound of the interval is Q_{M-1}. The use of an interval rather than a single degree is due to the degree Q_2. It is justified by the fact that the addition of one or some elements (i.e., a very weak quantity) to a set can either preserve its symbolic degree of proportion or increase it at most one degree.

Definition 7 *The symbolic sum* **S** *is a commutative application of $\mathfrak{Q}_{\mathfrak{M}}^2$ into $\mathfrak{P}(\mathfrak{Q}_{\mathfrak{M}})$. By supposing that $\alpha + \beta \leq$ M+1,* **S** *is defined as follows:*

$$\mathrm{S}(Q_\alpha, Q_\beta) = \begin{cases} \{Q_\alpha\} \text{ if } \beta = 1 \\ [Q_{\alpha+\beta-2}, Q_{\alpha+\beta-1}] \text{ if } \\ \alpha \neq 1,\ \beta \neq 1,\ \alpha + \beta \leq M \\ \{Q_{M-1}\} \text{ if } \alpha + \beta = M + 1 \end{cases}$$

In agreement with Axiom 3, it is necessary to have $\alpha + \beta \leq$ M+1. Indeed, A∩B $=\emptyset$ implies that B$\subset \overline{A}$. Now ((Axiom 3) gives $\Omega \subset_{n(\alpha)} \overline{A}$. Intuitively $\beta \leq$n(α) (for, B$\subset \overline{A}$) therefore, $\alpha + \beta \leq \alpha$+n($\alpha$)=M+1. Defining **Inf** and **Sup** as two applications of $\mathfrak{Q}_{\mathfrak{M}}^2$ into $\mathfrak{Q}_{\mathfrak{M}}$, we obtain respectively the lower bound and the upper bound of an interval of Q_M so we can write : $\mathbf{S}(Q_\alpha, Q_\beta)$ = [Inf$\circ\mathbf{S}(Q_\alpha, Q_\beta)$, Sup$\circ\mathbf{S}(Q_\alpha, Q_\beta)$] or more simply [**InfS**(Q_α, Q_β), **SupS**(Q_α, Q_β)]. We can prove that the applications **InfS** and **SupS** verify the properties of a T-conorm.

Definition 8 *An application T of $\mathfrak{L}_{\mathfrak{M}}^2$ into $\mathfrak{L}_{\mathfrak{M}}$ satisfying the following conditions :*
- $T(\tau_1, \tau_1) = \tau_1$,
- $T(\tau_\beta, \tau_M) = \tau_\beta$,
- $T(\tau_\beta, \tau_\alpha) = T(\tau_\alpha, \tau_\beta)$
- $\tau_{\beta_1} \leq \tau_{\beta_2}$ and $\tau_{\alpha_1} \leq \tau_{\alpha_2} \Longrightarrow T(\tau_{\alpha_1}, \tau_{\beta_1}) \leq T(\tau_{\alpha_2}, \tau_{\beta_2})$
- $T(\tau_\alpha, T(\tau_\beta, \tau_\gamma)) = T(T(\tau_\alpha, \tau_\beta), \tau_\gamma)$ is called a **T-norm***.*
Moreover, $C(\tau_\alpha, \tau_\beta) = \sim T(\sim \tau_\alpha, \sim \tau_\beta)$ is called a **T-conorm***.*

Definition 9 *Given* **S***, we can define the "symbolic subtraction" denoted* **D** *as an application of $\mathfrak{Q}_{\mathfrak{M}}^2$ into $\mathfrak{P}(\mathfrak{Q}_{\mathfrak{M}})$ such that: if $Q_r \in S(Q_\alpha, Q_\beta)$, then Q_β*

$\in$**D**(Q_r,Q_α) *and* $Q_\alpha \in D(Q_r,Q_\beta)$. *Then* **D** *can be deduced from* **S**:

$$\mathbf{D}\ (Q_r,\ Q_\beta) = \begin{cases} \{Q_r\} \text{ if } \beta = 1 \\ \{Q_2\} \text{ if } r{=}\beta \in [2,\ M-1] \\ [Q_{r+1-\beta}, Q_{r+2-\beta}] \text{ if} \\ \ 2 \le \beta < r \le M-1 \end{cases}$$

Remark 2 *In this paper, we have chosen the operators S and D defined by Tables 3 and 4 (see Annex A)*

5 Fundamental properties

Let A and b be substes of Ω. The following properties can be viewed as symbolic generalizations of classical statistical probabilities.

Proposition 3 *If* $\Omega \subset_\alpha A$ *and* $A \subset B$, *then* $\Omega \subset_\beta B$ *with* $Q_\alpha \le Q_\beta$.

Propoposition 3 shows that the symbolic degree of proportion of a set is greater than or equal to its subsets. Classically, the proportion of a set is strictly greater than to one of its strict subsets, while qualitatively, a set and one of its subsets can have the same symbolic degree of proportion (*Cf.* Axiom 1).

Proposition 4 *If* $\Omega \subset_\alpha A$, $\Omega \subset_\lambda A\cap B$ *and* $A \neq \Omega$, *then* $\Omega \subset_\gamma A\backslash B$ *with* $Q_\gamma \in D(Q_\alpha,\ Q_\lambda)$.

It appears clearly that Proposition 4 generalize the property: $|A\backslash B|/|\ \Omega\ | = (|A|-|A\cap B|)/|\ \Omega\ |$.

Proposition 5 *If* $\Omega \subset_\alpha A$, $\Omega \subset_\beta B$, $\Omega \subset_\lambda A\cap B$ *and* $A\cup B \neq \Omega$, *then* $\Omega \subset_r A\cup B$ *with* $Q_r \in U(Q_\alpha,\ Q_\beta,\ Q_\lambda)$ *where* $U(Q_\alpha,\ Q_\beta,\ Q_\lambda) = [InfS(Q_\alpha,\ InfD(Q_\beta,\ Q_\lambda)),\ SupS(Q_\alpha,\ SupD(Q_\beta,\ Q_\lambda))]$ *if* $\alpha+\beta-\lambda \le M-1$, *and* $U(Q_\alpha,\ Q_\beta,\ Q_\lambda) = \{Q_{M-1}\}$ *if* $\alpha+\beta-\lambda = M$.

Corollary 6 *If* $\Omega \subset_\alpha A$, $\Omega \subset_\beta B$, $\Omega \subset_r A\cup B$ *and* $A\cup B \neq \Omega$, *then* $\Omega \subset_\lambda A\cap B$ *with* $Q_\lambda = Q_2$ *if* $\alpha+\beta-r = 1$ *and* $Q_\lambda \in [InfD(Q_\beta,\ SupD(Q_r,\ Q_\alpha)),\ Inf\{SupD(Q_\beta,\ InfD(Q_r,\ Q_\alpha)),Q_\alpha,\ Q_\beta\}]$ *otherwise.*

Proposition 5 and Corollary 6 generalize the classical property: $|A\cup B|\ /\ |\ \Omega\ | = (|A| + |B| - |A\cap B|)\ /\ |\ \Omega\ |$.

6 Inference with quantifiers

Reasoning on quantifiers is called by Zadeh (Zadeh 85) *syllogistic reasoning*, where a syllogism is an inference rule that consists in deducing a new quantified statement from one or several quantified statements. As an inference scheme, a *syllogism* may generally be expressed in the form:

$Q_{\mu 1}$ A's are B's
$Q_{\mu 2}$ C's are D's

Q_μ E's are F's with $Q_\mu \in [Q_a,\ Q_b] \subseteq [Q_1,\ Q_M]$,
where E and F are sets resulting from applications of set operators on A, B, C or D, and where the bounds Q_a and Q_b are in accordance with $Q_{\mu 1}$ and $Q_{\mu 2}$.

The quantifier "All" is represented by the implication using the quantifier $\forall$ in classical logic or by the inclusion in set theory. The classical implication and the inclusion propagate inferences by transitivity, contraposition, disjunction or by conjunction. From one or several statements quantified by "All", these inferences enable to generate new statements likely quantified by "All". Nevertheless, most of these inferences are not valid for other quantifiers, i.e., for $Q_\mu \in [Q_2, Q_{M-1}]$. For example, from "Most A's are B's" and "Most B's are C's" one can not always have "Most A's are C's". That is due to the fact that the inference by transitivity is not valid for the quantifier "Most". The invalid inference has been considered as a case of total ignorance.

6.1 Valid inferences with quantifiers

We consider that an inference is **valid**, if we deduce $Q_\mu \in [Q_a,\ Q_b]$, where Q_a or Q_b is in accordance with $Q_{\mu 1}$ or $Q_{\mu 2}$. We present some valid inferences. Each of them is illustrated by an example.

Proposition 7 *(Relative Duality):*
$Q_{\mu 1}$ *A's are B's*

$Q_{\mu 2}$ *A's are* $A\backslash B$ *and* $Q_{\mu 2}$ *A's are not B's*
with $Q_{\mu 2} = Q_{n(\mu 1)}$ *if* $Q_{\mu 1} \neq Q_{n(\mu 1)}$
and $Q_{\mu 2} \in [Q_{n(\mu 1)},\ Q_{n(\mu 1)+1}]$ *otherwise.*

Example 3
Almost all students are unmarried

Very few students are married.

Proposition 8 *(Mixed Transitivity):*
$Q_{\mu 1}$ *A's are B's*
All B's are C's

$Q_{\mu 2}$ *A's are C's with* $Q_{\mu 1} \le Q_2$

Example 4
Most students are young (less than 25 years)
All young people are non retired

At least most students are non retired.

Proposition 9 *(Exception)*
Q_μ *A's are B's*
All C's are A's
All C's are not B

Q_γ *A's are not C, with* $Q_\gamma \in [Q_\mu, Q_{M-1}]$.

Example 5
Most birds fly
All ostriches are birds
All ostriches do not fly

Most or almost all birds are not ostriches.

Proposition 10 *(Union Right)*
$Q_{\mu 1}$ *A's are B's*
$Q_{\mu 2}$ *A 's are C's*

Q_μ *A's are* $(B \cup C)$*'s, with* $Q_\mu \in [Q_{Max(\mu 1,\mu 2)}, Q_{M-1}]$.

Example 6
Most students are single
Very few students are taxable

Most or almost all students are single or taxable.

Proposition 11 *(Intersection Right)*
$Q_{\mu 1}$ *A's are B's*
$Q_{\mu 2}$ *A 's are C's*
———————
Q_μ *A's are* $(B \cap C)$*'s, with* $Q_\mu \leq Q_{Min(\mu 1,\mu 2)}$

Example 7
Few salaried people are official
Most salaried people are taxable
——————————————
At most salaried people are taxable official.

Proposition 12 *(Mixed Union Left)*
Q_μ *A's are C's*
All B's are C's
———————
Q_γ $(A \cup B)$*'s are C's with* $Q_\gamma \in [Q_\mu, Q_{M-1}]$

Example 8 ***Example 9***
Most young people are single
All the catholic priests are single
———————————————
Most or almost all young people or catholic priests are single.

Proposition 13 *(Intersection / Product Syllogism):*
$Q_{\mu 1}$ *A's are B's*
$Q_{\mu 2}$ $(A \cap B)$*'s are C's*
———————————————
Q_μ *A's are* $(B \cap C)$*'s, with* $Q_\mu = I(Q_{\mu 1}, Q_{\mu 2})$

Example 10
Most students are young
Almost all young students are unmarried
———————————————
Most students are young and unmarried.

Proposition 14 *(Contraction)*
$Q_{\mu 1}$ *A's are B's*
$Q_{\mu 2}$ $(A \cap B)$*'s are C's*
————————
Q_μ *A's are C's, with* $Q_\mu = [I(Q_{\mu 1}, Q_{\mu 2}), Q_{Max(M-1,\mu 2)}]$

Example 11
Most students are young
Almost all young students are unmarried
——————————————————
Most or almost all students are young and unmarrie

Proposition 15 *(Intersection/Quotient syllogism)*
$Q_{\mu 1}$ *A's are B's*
$Q_{\mu 2}$ *A 's are C's*
$Q_{\mu 3}$ $(A \cap B)$ *'s are C's*
————————
Q_μ $(A \cap C)$*'s are B's, with* $Q_\mu \in C(Q_{\mu 2}, I(Q_{\mu 1}, Q_{\mu 3}))$.

Example 12
Most students are young
Most students non salaried
Almost all young students are non salaried
———————————————————
Almost all non-salaried students are young.

Proposition 16 *(Weak Transitivity)*
All B's are A's
$Q_{\mu 1}$ *A 's are C's*
$Q_{\mu 2}$ *B's are C's*
——————
Q_μ *A's are C's with* $Q_\mu \in [I(Q_{\mu 1}, Q_{\mu 2}), \textit{Almost-all}]$.

Example 13
All salaried people are active
Most active people are salaried
Most salaried people are taxable
———————————————
Between half and almost all of active people are taxable

6.2 Valid Inferences with the quantifier "Almost-all"

We present three inferences only valid with the quantifier "Almost-all". They result from the axioms of quantifier "Almost-all" (*Cf.* Axiom 1, Axiom 2). These inferences are counterparts [7] of inference rules proposed in ((Adams 75), (Pearl 91)), where "Almost all" is interpreted as proportion arbitrarily close to 1. In their approaches, the obtained quantifier is "Almost-all". In our approach, with the first inference we obtain "Almost-all", while with the others, we obtain "Most" or "Almost-all". That can be justified by the fact that we consider the infinitesimal interpretation of this quantifier is a particular case of our interpretation.

Proposition 17 *(Contraction)*
Almost-all A's are B's
Almost-all $(A \cap B)$*'s are C's*
————————————
Almost-all A's are C's.

Example 14
Almost all students are young
Almost all young students are single
———————————————
Almost all students are single.

Proposition 18 *(Cumulativity)*
Almost-all A's are B's
Almost-all A's are C's.
————————————
Q_μ $(A \cap B)$*'s are C's, with* $Q_\mu \in [\textit{Most, Almost-all}]$.

Example 15
Almost all students are young
Almost all students are single
————————————
Most or almost all young students are single.

Proposition 19 *(Union Left)*
Almost-all A's are C's
Almost-all B's are C's.
——————————

[7] Pearl's approach is introduced for default reasoning, then his inferences are not exactly syllogisms, but they are rather non-monotonic inferences about particular individuals from defaults.

Q_μ *(A* $\cup$ *B)'s are C's, with* $Q_\mu \in$*[Most, Almost-all].*

Example 16
Almost all students are single
Almost all priests are single
——————————————
Most or almost all students or priests are single.

6.3 Monotonic aspect of reasoning with quantifiers

We can note that the reasoning with quantifiers is monotonic (Sombé 90) in the following sense: when a knowledge base contains: $A \subset_{\mu 1} B$ and $C \subset_{\mu 2} D$, when we deduce $E \subset_\mu F$ with $Q_\mu \in [Q_a, Q_b]$, and if one adds in the base or one deduces by an other inference new information: $E \subset_{\mu'} F$ with $Q_{\mu'} \in [Q_{a'}, Q_{b'}]$, then one must have $[Q_a, Q_b] \cap [Q_{a'}, Q_{b'}] \neq \emptyset$ and finally one will have: $E \subset_\mu F$ with $Q_\mu \in [Q_a, Q_b] \cap [Q_{a'}, Q_{b'}]$. In other words, the new knowledge can only tighten the interval $[Q_a, Q_b]$, that maintains the coherence between the quantified statements. There is an *inconsistency*, if $[Q_a, Q_b] \cap [Q_{a'}, Q_{b'}] = \emptyset$.

Example 17 *The following example proposed among benchmarks problems contains the quantified statements:*
S1: Most dancers are not ballerinas,
S2: Most dancers are graceful,
S3: Most graceful dancers are ballerinas.
Then, S1 $\Longrightarrow$ *"Few dancers are ballerinas" (Relative duality).*
Moreover, S1 and S2 $\Longrightarrow$ *"Between half and almost all dancers are ballerinas" (Contraction).*
Since $\{Few\} \cap$ *[Half,Almost-all]* $= \emptyset$*, the statements S1, S2 and S3 are inconsistent.*

7 Conclusion

In this paper we have presented a symbolic approach to quantifiers used in the natural language to express a qualitative evaluation of proportions. This approach allows to reason qualitatively on quantified assertions, since we provide inference rules based upon statements involving linguistic quantifiers. In our interpretation, in accordance with the natural language, the proposition "Q A's are B's" have a Boolean nature, i.e., it is given as true. On other hand, in fuzzy approaches, the proposition is multivalent, in the sense, that for a given proportion r of B in A, the truth degree of "Q A's are B's" is equal to $\mu_Q(r)$. The difficulty of choice of the inteval bounds values and of membership functions in semi-numeric approaches does not face in our approach purely symbolic. Moreover, in the approaches based upon the cardinality of sets to represent proportional quantifiers, the sets must be finite, while in our symbolic approach, they can be infinite. Thus, we can represent statements like "Most birds fly". In order to obtain belief symbolic degrees attached to properties about *particular individuals*, and this, by using knowledge based upon quantified assertions and certain facts, it is nececarry to propose a symbolic model based upon a *direct inference principle* and a choice of the appropriated *reference class* ((Kyburg 83), (Pollock 90), (Bacchus 90)). This point is actually on study.

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Annex A : Tables of operators C, I, S and D

Remark 3 *In the following tables,* $Q_{a,b}$ *stands for interval [*Q_a*,* Q_b*]*

C	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
Q_1	$Q_{1,7}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Q_2	$\{Q_1\}$	$Q_{2,7}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Q_3	$\{Q_1\}$	$Q_{2,5}$	$Q_{6,7}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Q_4	$\{Q_1\}$	$Q_{2,4}$	$\{Q_5\}$	$Q_{6,7}$	$\emptyset$	$\emptyset$	$\emptyset$
Q_5	$\{Q_1\}$	$Q_{2,3}$	$\{Q_4\}$	$\{Q_5\}$	$Q_{6,7}$	$\emptyset$	$\emptyset$
Q_6	$\{Q_1\}$	$\{Q_2\}$	$\{Q_3\}$	$\{Q_4\}$	$\{Q_5\}$	$Q_{6,7}$	$\emptyset$
Q_7	$\{Q_1\}$	$\{Q_2\}$	$\{Q_3\}$	$\{Q_4\}$	$\{Q_5\}$	$\{Q_6\}$	$\{Q_7\}$

Table 1 : Operator C

I	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
Q_1	Q_1	Q_1	Q_1	Q_1	Q_1	Q_1	Q_1
Q_2	Q_1	Q_2	Q_2	Q_2	Q_2	Q_2	Q_2
Q_3	Q_1	Q_2	Q_2	Q_2	Q_2	Q_3	Q_3
Q_4	Q_1	Q_2	Q_2	Q_2	Q_3	Q_4	Q_4
Q_5	Q_1	Q_2	Q_2	Q_3	Q_4	Q_5	Q_5
Q_6	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_6
Q_7	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7

Table 2 : Operator I

S	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
Q_1	$\{Q_1\}$	$\{Q_2\}$	$\{Q_3\}$	$\{Q_4\}$	$\{Q_5\}$	$\{Q_6\}$
Q_2	$\{Q_2\}$	$Q_{2,3}$	$Q_{3,4}$	$Q_{4,5}$	$Q_{5,6}]$	$\{Q_6\}$
Q_3	$\{Q_3\}$	$Q_{3,4}$	$Q_{4,5}$	$Q_{4,6}$	$\{Q_6\}$	$\{Q_3\}$
Q_4	$\{Q_4\}$	$Q_{4,5}$	$Q_{5,6}$	$\{Q_6\}$	$\{Q_3\}$	$\{Q_4\}$
Q_5	$\{Q_5\}$	$Q_{5,6}$	$\{Q_6\}$	$\{Q_3$	$\{Q_4\}$	$\{Q_5\}$
Q_6	$\{Q_6\}$	$\{Q_6\}$	$\{Q_3$	$\{Q_4$	$\{Q_5\}$	$\{Q_6\}$

Table 3 : Operator S

D	Q_1	Q_2	Q3	Q4	Q5	Q6
Q_1	$\{Q_1\}$					
Q_2	$\{Q_2\}$	$\{Q_2\}$				
Q_3	$\{Q_3\}$	$Q_{2,3}$	$\{Q_2\}$			
Q_4	$\{Q_4\}$	$Q_{3,4}$	$Q_{2,3}$	$\{Q_2\}$		
Q_5	$\{Q_5\}$	$Q_{4,5}$	$Q_{3,4}$	$Q_{2,3}$	$\{Q_2\}$	
Q_6	$\{Q_6\}$	$Q_{5,6}$	$Q_{4,5}$	$Q_{3,4}$	$Q_{2,3}$	$\{Q_2\}$

Table 4 :Operator D

Annex B : Proofs of some propositions

We suppose that A, B and C are subsets of Ω .

Proposition 3: Let us suppose that $\Omega \subset_\beta$B and $\Omega \subset_{\beta'}$B\A. A $\subset$B $\Longrightarrow$ B = A$\cup$ B\A. A and A\B are disjoint. Using Axiom 4, we obtain: $Q_\beta \in S(Q_\alpha, Q_{\beta'})$. Therefore $Q_\alpha \leq Q_\beta$.

Proposition 4: Let us suppose that $\Omega \subset_{\alpha'}$A\B. A = (A $\cap$B) $\cup$A\B and (A$\cap$ B) $\cap$ A\B = $\emptyset$. Using Axiom 4, we obtain: $Q_\alpha \in S(Q_\lambda, Q_{\alpha'})$. Definition 8 gives us : $Q_{\alpha'} \in D(Q\alpha, Q_\lambda)$.

Proposition 5: Let us suppose that $\Omega \subset_{\beta'}$B\A. The proposition 4 gives us : $Q_{\beta'} \in$[InfD(Q_β, Q_λ), SupD(Q_β, Q_λ)]. Since A$\cup$B=A $\cup$ (B \ A) and the sets A and B\A are disjoint, then according to Axiom 4 : if $\alpha+\beta' \leq$ M+1, $\Omega \subset_r$A$\cup$B with $Q_r \in S(Q_\alpha, Q_{\beta'})$. Definition 8 implies that : InfD(Q_β, Q_λ) = $Q_{\beta+1-\lambda} \leq Q_{\beta'} \leq$ SupDλ(Q_β, Q_λ) = $Q_{\beta+2-\lambda}$. Therefore, $\alpha+\beta' \leq$ M+1 $\Longrightarrow \alpha+\beta+1-\lambda \leq \alpha+\beta' \leq \alpha+\beta+2 - \lambda \leq$ M+1. So, we obtain (see Table 5):

- If $\alpha+\beta+2 - \lambda \leq$ M+1 $\Longrightarrow \alpha+\beta+ \lambda \leq$ M-1, then $Q_r \in$[InfS(Q_α, InfD(Q_β, Q_λ)), SupS(Q_α, SupD(Q_β,Q_λ))].
- If $\alpha+\beta+1 - \lambda$ = M+1 $\Longrightarrow \alpha+\beta- \lambda \leq$ M, then $Q_r \in$ {InfS(Q_α, InfD(Q_β, Q_λ))} = {Q_{M-1}}.

Q_α	Q_β	Q_λ	$Q_r \in U(Q_\alpha, Q_\beta, Q_\lambda)$
Q_2	Q_2	Q_2	$[Q_2,Q_3]$
Q_3	Q_2	Q_2	$[Q_3,Q_4]$
Q_3	Q_3	Q_2	$[Q_3,Q_5]$
Q_3	Q_3	Q_3	$[Q_3,Q_4]$
Q_4	Q_2	Q_2	$[Q_4,Q_5]$
Q_4	Q_3	Q_2	$[Q_4,Q_6]$
Q_4	Q_3	Q_3	$[Q_4,Q_5]$
Q_4	Q_4	Q_2	$[Q_5,Q_6]$
Q_4	Q_4	Q_3	$[Q_4,Q_6]$
Q_4	Q_4	Q_4	$[Q_4,Q_5]$
Q_5	Q_2	Q_2	$[Q_5,Q_6]$
Q_5	Q_3	Q_2	$[Q_5,Q_6]$
Q_5	Q_3	Q_3	$[Q_5,Q_6]$
Q_5	Q_4	Q_2	$\{Q_6\}$
Q_5	Q_4	Q_3	$[Q_5,Q_6]$
Q_5	Q_4	Q_4	$[Q_5,Q_6]$
Q_5	Q_5	Q_3	$\{Q_6\}$
Q_5	Q_5	Q_4	$[Q_5,Q_6]$
Q_5	Q_5	Q_5	$[Q_5,Q_6]$
Q_6	Q_2	Q_2	$\{Q_6\}$
Q_6	Q_3	Q_2	$\{Q_6\}$
Q_6	Q_3	Q_3	$\{Q_6\}$
Q_6	Q_4	Q_3	$\{Q_6\}$
Q_6	Q_4	Q_4	$\{Q_6\}$
Q_6	Q_5	Q_4	$\{Q_6\}$
Q_6	Q_5	Q_5	$\{Q_6\}$
Q_6	Q_6	Q_5	$\{Q_6\}$
Q_6	Q_6	Q_6	$\{Q_6\}$

Table 5: $U(Q_\alpha, Q_\beta, Q_\lambda)$

Corollary 6 : Let us suppose $\Omega \subset_\lambda$A$\cap$ B. Proposition 5 give us $Q_r \in$U(Q_α, Qβ, Qλ) if $\alpha + \beta - \lambda \leq$ M - 1, and $Q_r \in \{Q_{M-1}\}$ if $\alpha + \beta - \lambda$= M. Thus we can deduce the values of Q_λ in accordance with those of Q_r, Q_α and Q_β, as this is showed in the following table 6. From this table, we can verify that : $Q_\lambda = Q_2$ if $\alpha + \beta - r$ =1 and $Q_\lambda \in$[InfD(Q_β, SupD(Q_r, Q_α)), Inf{SupD(Q_β, InfD(Q_r, Q_α)), Q_α, Q_β}] otherwise.

Q_r	Q_α	Q_β	$Q_\lambda \in$
Q_2	Q_2	Q_2	$\{Q_2\}$
Q_3	Q_2	Q_2	$\{Q_2\}$
Q_3	Q_3	Q_2	$\{Q_2\}$
Q_3	Q_3	Q_3	$[Q_2,Q_3]$
Q_4	Q_3	Q_2	$\{Q_2\}$
Q_4	Q_3	Q_3	$[Q_2,Q_3]$
Q_4	Q_4	Q_2	$\{Q_2\}$
Q_4	Q_4	Q_3	$[Q_2,Q_3]$
Q_4	Q_4	Q_4	$[Q_3,Q_4]$
Q_5	Q_3	Q_3	$\{Q_2\}$
Q_5	Q_4	Q_2	$\{Q_2\}$
Q_5	Q_4	Q_3	$[Q_2,Q_3]$
Q_5	Q_4	Q_4	$[Q_2,Q_4]$
Q_5	Q_5	Q_2	$\{Q_2\}$
Q_5	Q_5	Q_3	$[Q_2,Q_3]$
Q_5	Q_5	Q_4	$[Q_3,Q_4]$
Q_5	Q_5	Q_5	$[Q_4,Q_5]$
Q_6	Q_4	Q_3	$\{Q_2\}$
Q_6	Q_4	Q_4	$[Q_2,Q_3]$
Q_6	Q_5	Q_2	$\{Q_2\}$
Q_6	Q_5	Q_3	$[Q_2,Q_3]$
Q_6	Q_5	Q_4	$[Q_2,Q_4]$
Q_6	Q_5	Q_5	$[Q_3,Q_5]$
Q_6	Q_6	Q_3	$[Q_2,Q_3]$
Q_6	Q_6	Q_4	$[Q_3,Q_4]$
Q_6	Q_6	Q_5	$[Q_4,Q_5]$
Q_6	Q_6	Q_6	$[Q_5,Q_6]$

Table 6

Proposition 7 (Relative Duality) : Let us suppose that $\Omega \subset_\alpha$A, $\Omega \subset_{\lambda1}A\cap$ B and A$\subset_{\mu2}$ A\B. We have $Q_{\mu1}$ $\in$C(Q_α, $Q_{\lambda1}$) or equivalently $Q_{\lambda1}$ = I(Q_α, $Q_{\mu1}$). Using proposition 4, we obtain $\Omega \subset_{\lambda2}$A\B with $Q_{\lambda2}$ $\in$D(Q_α, $Q_{\lambda1}$). For the different degrees of Q_α and $Q_{\mu1}$, in $\mathfrak{L}_7$, the following table 7 gives us the values of $Q_{\mu2}$ $\in$C(Q_α, $Q_{\lambda2}$). We can verify that : $Q_{n(\mu1)+1}$ = Q_5 $\in$C(Q_α, $Q_{\lambda2}$), if Q_α = Q_4 and $Q_{\mu1}$ = Q_4. For the other cases, we can verify that : for any $Q_{\mu1}$ $\in$C(Q_α, $Q_{\lambda1}$), there exists $Q_{\mu2}$ $\in$C(Q_α, $Q_{\lambda2}$) such that $Q_{\mu2}$ = $Q_{n(\mu1)}$. Since A\B = A $\cap$ $\overline{B}$, then A$\subset_{\mu2}$ A$\cap\overline{B}$ and consequently A$\subset_{\mu2}$ $\overline{B}$.

Q_α	$Q_{\mu1} \in$	a	b	c
Q_2	$[Q_2,Q_6]$	Q_2	$\{Q_2\}$	$[Q_2,Q_6]$
Q_3	$[Q_2,Q_5]$	Q_2	$[Q_2,Q_3]$	$[Q_2,Q_6]$
Q_3	$\{Q_6\}$	Q_3	$\{Q_2\}$	$[Q_2,Q_5]$
Q_4	$[Q_2,Q_4]$	Q_2	$[Q_3,Q_4]$	$[Q_5,Q_6]$
Q_4	$\{Q_5\}$	Q_3	$[Q_2,Q_3]$	$[Q_2,Q_5]$
Q_4	$\{Q_6\}$	Q_4	$\{Q_2\}$	$[Q_2,Q_4]$
Q_5	$[Q_2,Q_3]$	Q_2	$[Q_4,Q_5]$	$[Q_5,Q_6]$
Q_5	$\{Q_4\}$	Q_3	$[Q_3,Q_4]$	$[Q_4,Q_5]$
Q_5	$\{Q_5\}$	Q_4	$[Q_2,Q_3]$	$[Q_2,Q_4]$
Q_5	$\{Q_6\}$	Q_5	$\{Q_2\}$	$[Q_2,Q_3]$
Q_6	$\{Q_2\}$	Q_2	$[Q_5,Q_6]$	$[Q_5,Q_6]$
Q_6	$\{Q_3\}$	Q_3	$[Q_4,Q_5]$	$[Q_4,Q_5]$
Q_6	$\{Q_4\}$	Q_4	$[Q_3,Q_4]$	$[Q_3,Q_4]$
Q_6	$\{Q_5\}$	Q_5	$[Q_2,Q_3]$	$[Q_2,Q_3]$
Q_6	$\{Q_6\}$	Q_6	$\{Q_2\}$	$\{Q_2\}$

a: $Q_{\lambda1} = I(Q_\alpha, Q_{\mu1})$

b: $Q_{\lambda2} \in D(Q_\alpha, Q_{\lambda1})$

c: $Q_{\mu2} \in C(Q_\alpha, Q_{\lambda2})\backslash\{Q_7\}$

Table 7

Proposition 8 (Mixed Transitivity) : Let us suppose $\Omega \subset_\alpha$A,$\Omega \subset_{\lambda1}$ A$\cap$ B, $\Omega \subset_{\lambda2}$A $\cap$C and A $\subset_{\mu2}$C. The definition 6 give us $Q_{\mu1}$ $\in$C(Q_α, $Q_{\lambda1}$) and $Q_{\mu2}$ $\in$C(Q_α, $Q_{\lambda2}$). B$\subset$ C $\Longrightarrow$ A$\cap$ B $\subset$ A $\cap$C. Using Proposition 3, we obtain $Q_{\lambda1}$ $\leq$$Q_{\lambda2}$. We distinguish two cases :
-a- When $Q_{\lambda1}$ < $Q_{\lambda2}$, Table 1 of operator C implies that $Q_{\mu1}$ < $Q_{\mu2}$.
-b- When $Q_{\lambda1}$ = $Q_{\lambda2}$, we distinguish three cases:
-b1- if A$\cap$ B = A $\cap$C, then $Q_{\mu1}$ = $Q_{\mu2}$.
-b2- if A $\cap$B $\neq$A$\cap$ C = A, then Proposition 1 implies : $Q_{\mu1}$ < Q_M and $Q_{\mu2}$ = Q_M. Therefore $Q_{\mu1}$ < $Q_{\mu2}$.
-b3- if A$\cap$B$\neq$ A $\cap$C$\neq$ A, then:
- when $Q_{\lambda1}$ $\geq$$Q_3$: Axiom 1 gives us $Q_{\mu1}$ = $Q_{\mu2}$ = Q_{M-1} if Q_α = $Q_{\lambda1}$, and we have : Card(C(Q_α, $Q_{\lambda1}$)) = 1 otherwise (i.e., Q_α $\neq$$Q_{\lambda1}$ = $Q_{\lambda2}$). Therefore $Q_{\mu1}$ = $Q_{\lambda2}$.
- when $Q_{\lambda1}$ = Q_2 : since we have C(Q_α, $Q_{\lambda1}$) = C(Q_α, $Q_{\lambda2}$), then for any $Q_{\mu1}$ $\in$C(Q_α, $Q_{\lambda1}$) there exists $Q_{\mu2}$ $\in$C(Q_α, $Q_{\lambda2}$) such that $Q_{\mu1}$ $\leq$$Q_{\mu2}$.

Proposition 13 (Intersection / product Syllogism) : Let us suppose that $\Omega \subset_\alpha$A, $\Omega \subset_\lambda$A$\cap$ B, $\Omega \subset_\delta$A $\cap$B$\cap$ C and A $\subset_\mu$B $\cap$C. We have Q_λ = I(Q_α, $Q_{\mu1}$), Q_δ = I(Q_λ, $Q_{\mu2}$) and Q_μ $\in$C(Q_α, Q_δ). >From the following table 8 in $\mathfrak{L}_7$, we can verify that for the different degrees of Q_α, Q_λ, Q_δ, $Q_{\mu1}$ and $Q_{\mu2}$ with Q_α $\geq$$Q_\lambda$ $\geq$$Q_\beta$ we have :
- if Q_δ $\geq$$Q_3$, then Q_μ = I($Q_{\mu1}$, $Q_{\mu2}$).
- if Q_δ = Q_2, then C(Q_α, Q_δ) = [I(InfC(Q_α, Q_λ), InfC(Q_λ, Q_δ)), I(SupC(Q_α, Q_λ), SupC(Q_λ, Q_δ))]. Since for any $Q_{\mu1}$ $\in$C(Q_α, Q_λ) and $Q_{\mu2}$ $\in$C(Q_λ, Q_δ), we have I($Q_{\mu1}$, $Q_{\mu2}$) $\in$C(Q_α, Q_δ). Therefore Q_μ = I($Q_{\mu1}$, $Q_{\mu2}$).

Q_α	Q_λ	Q_δ	a	b	c	d
Q_2	Q_2	Q_2	$[Q_2,Q_6]$	$[Q_2,Q_6]$	$[Q_2,Q_6]$	$[Q_2,Q_6]$
Q_3	Q_2	Q_2	$[Q_2,Q_5]$	$[Q_2,Q_6]$	$[Q_2,Q_5]$	$[Q_2,Q_5]$
Q_3	Q_3	Q_2	$\{Q_6\}$	$[Q_2,Q_5]$	$[Q_2,Q_5]$	$[Q_2,Q_5]$
Q_4	Q_3	Q_2	$\{Q_5\}$	$[Q_2,Q_5]$	$[Q_2,Q_4]$	$[Q_2,Q_4]$
Q_4	Q_4	Q_2	$\{Q_6\}$	$[Q_2,Q_4]$	$[Q_2,Q_4]$	$[Q_2,Q_4]$
Q_5	Q_3	Q_2	$\{Q_4\}$	$[Q_2,Q_5]$	$[Q_2,Q_3]$	$[Q_2,Q_3]$
Q_5	Q_4	Q_2	$\{Q_5\}$	$[Q_2,Q_4]$	$[Q_2,Q_3]$	$[Q_2,Q_3]$
Q_6	Q_4	Q_2	$\{Q_4\}$	$[Q_2,Q_4]$	$\{Q2\}$	$\{Q2\}$
Q_6	Q_6	Q_2	$\{Q_6\}$	$\{Q2\}$	$\{Q2\}$	$\{Q2\}$
Q_3	Q_3	Q_3	$\{Q_6\}$	$\{Q_6\}$	$\{Q_6\}$	$\{Q_6\}$
Q_4	Q_3	Q_3	$\{Q_5\}$	$\{Q_6\}$	$\{Q_5\}$	$\{Q_5\}$
Q_4	Q_4	Q_4	$\{Q_6\}$	$\{Q_6\}$	$\{Q_6\}$	$\{Q_6\}$
Q_5	Q_4	Q_3	$\{Q_5\}$	$\{Q_5\}$	$\{Q_4\}$	$\{Q_4\}$
Q_5	Q_4	Q_4	$\{Q_5\}$	$\{Q_6\}$	$\{Q_5\}$	$\{Q_5\}$
Q_5	Q_5	Q_5	$\{Q_6\}$	$\{Q_6\}$	$\{Q_6\}$	$\{Q_6\}$
Q_6	Q_5	Q_4	$\{Q_5\}$	$\{Q_5\}$	$\{Q_4\}$	$\{Q_4\}$
Q_6	Q_4	Q_5	$\{Q_5\}$	$\{Q_6\}$	$\{Q_5\}$	$\{Q_5\}$
Q_6	Q_5	Q_4	$\{Q_6\}$	$\{Q_5\}$	$\{Q_5\}$	$\{Q_5\}$
Q_6	Q_6	Q_5	$\{Q_6\}$	$\{Q6\}$	$\{Q_6\}$	$\{Q_6\}$

a: $Q_{\mu 1} \in C(Q_\alpha, Q_\lambda)\backslash\{Q_7\}$
b: $Q_{\mu 2} \in C(Q_\lambda, Q_\delta)\backslash\{Q_7\}$
c: $I(Q_{\mu 1}, Q_{\mu 2})$
d: $Q_\mu \in C(Q_\alpha, Q_\delta)\backslash\{Q_7\}$

Table 8

Proposition 18 (Cumulativity) : Let us suppose that $\Omega \subset_\alpha$A, $\Omega \subset_\lambda$A$\cap$ B$\cap$C and A$\cap$ B$\subset_\mu$ C. We have $Q_\mu \in C(Q_\alpha, Q_\lambda)$. Axiom 1 implies that : (A$\subset_{M-1}$ B $\Longrightarrow \Omega \subset_\alpha$A $\cap$B) and (A $\subset_{M-1}$ C $\Longrightarrow \Omega \subset_\alpha$A $\cap$C) and Proposition 1 implies that A$\cap$ B$\neq$ A and A$\cap$ C $\neq$A. Since A $\cap$ B $\subset$ (A$\cap$ C) $\cup$ (B$\cap$ C) $\subset$ A, $\Omega \subset_\alpha$A and $\Omega \subset_\alpha$A $\cap$B, then $\Omega \subset_\alpha$(A $\cap$C) $\cup$(B $\cap$C). Corollary 6 give us $Q_\lambda \in$[InfD(Q_α, SupD(Q_α, Q_α)), SupD(Q_α, InfD(Q_α, Q_α))] which is equal to $[Q_{\alpha-1},Q_\alpha]$ for $\alpha \geq 3$ and to $\{Q_2\}$ for $\alpha = 2$. Since A$\cap$ B $\neq$A and A $\cap$C$\neq$A, then A $\cap$B $\cap$C $\neq$ A$\cap$ B. Therefore $Q_\mu \in C(Q_\alpha, Q_\lambda)\backslash\{Q_M\}$ which is equal to ([InfC(Q_α, $Q_{\alpha-1}$), SupC(Q_α, Q_α)]) $\backslash$ $\{Q_M\}$ for $\alpha \geq 3$ and to $C(Q_2, Q_2)\backslash\{Q_M\}$ for α=2. It is evident to verify in table 1 of C that : for $\alpha \geq$4, [InfC(Q_α, $Q_{\alpha-1}$), SupC(Q_α, Q_α] $\backslash$ $\{Q_M\}$ = $[Q_{M-2},Q_{M-1}]$; for $2 \leq \alpha \leq 3$, $C(Q_\alpha, Q_\lambda)\backslash\{Q_M\}$ = $[Q_2, Q_{M-1}]$. Since for $\alpha \geq 4$, we have $Q_\mu \in [Q_{M-2}, Q_{M-1}]$, then for $2 \leq \alpha \leq 3$ the interval $[Q_2,Q_{M-1}]$ can be restrained to the interval $[Q_{M-2},Q_{M-1}]$. Therefore $Q\mu \in [Q_{M-2},Q_{M-1}]$.

Proposition 17 (Contraction) : According to Proposition 9 : A$\subset_{M-1}$ B and A $\cap$B$\subset_{M-1}$C $\Longrightarrow$A$\subset_\gamma$ B $\cap$C with $Q_\gamma = I(Q_{M-1}, Q_{M-1}) = Q_{M-1}$. According to Proposition 8 : A$\subset_\gamma$ B $\cap$C and B $\cap$C $\subset$C $\Longrightarrow$A $\subset_\mu$C with $Q_\mu \in [Q_{M-1},Q_M]$. A$\cap$ B $\subset_{M-1}$ C $\Longrightarrow Q_\mu < Q_M$. Thus $Q_\mu = Q_{M-1}$.

Proposition 19 (Union Left) : Let us suppose that $\Omega \subset_\alpha$A, $\Omega \subset_\beta$B, $\Omega \subset_{\lambda 1}$A $\cap$B and $\Omega \subset_{\lambda 2}A\cap$B $\cap$C. According to Axiom 2 : (A$\subset_{M-1}$ C and $\Omega \subset_\alpha$A $\Longrightarrow$ $\Omega \subset_\alpha$A$\cap$ C) and (B $\subset_{M-1}$ C and $\Omega \subset_\beta$B$\Longrightarrow \Omega \subset_\beta$B$\cap$ C). Proposition 1 implies that A $\cap$C $\neq$A and B$\cap$ C$\neq$ B. We have (A $\cup$B)$\cap$C = (A$\cap$ C) $\cup$(B$\cap$ C)). Then, Proposition 5 gives us :

- $\Omega \subset_{r1}$A$\cup$ B with $Q_{r1} \in U(Q, Q, Q)$ = [InfS(Q_α, InfD(Q_β, $Q_{\lambda 1}$)), SupS(Q_α, SupD(Q_β, $Q_{\lambda 1}$))] if α + β - λ1 $\leq$ M-1, and $Q_{r1} = Q_{M-1}$if α + β - λ1= M ;
- $\Omega \subset_{r2}$ (A $\cap$C) $\cup$(B $\cap$C) with $Q_{r2} \in U(Q_\alpha, Q_\beta, Q_{\lambda 2})$ = [InfS(Q_α, InfD(Q_β, $Q_{\lambda 2}$)), SupS(Q_α, SupD(Q_β, $Q_{\lambda 2}$))] if α + β - λ2$\leq$ M-1, and Q_{r2} =Q_{M-1} if α + β - λ2 = M.

Since (A$\cup$ B) $\cap$C$\subset$A$\cup$B and A$\cap$ B $\cap$C $\subset$A $\cap$B, then Proposition 3 gives us : $Q_{r2} \leq Q_{r1}$ and $Q_{\lambda 2} \leq Q_{\lambda 1}$.
The following constraint : for any $Q_{\lambda 2}$, $Q_{\lambda 1}$ such that $Q_{\lambda 2} \leq Q_{\lambda 1}$ we must have $Q_{r2} \leq Q_{r1}$, allow us to suppress each value of U(Q_α, Q_β, $Q_{\lambda 1}$) and U(Q_α, Q_β, $Q_{\lambda 2}$) for which $Q_{r2} > Q_{r1}$ as that is showed in the following table 9. From this table, we can verify that for any Q_α, Q_β, $Q_{\lambda 2}$, Q_λ with $Q_{\lambda 2} \leq Q_{\lambda 1}$, we have Q_{r2} = Q_{r1} or $Q_{r2} = Q_{r1-1}$.
If A$\cup$B$\subset_\mu$C, then $Q_\mu \in C(Q_{r1},Q_{r2})$. We have (A$\cup$B) $\cap$ C = (A$\cap$C) $\cup$ (B$\cap$C) $\neq$ A$\cup$B (as,A$\cap$C$\neq$A and B$\cap$C$\neq$B), then $Q_\mu \in C(Q_{r1}, Q_{r2}) \backslash \{Q_M\}$ which is equal to ([InfC(Q_{r1}, Q_{r1-1}), SupC(Q_{r1}, Q_{r1})]) $\backslash$ $\{Q_M\}$ for $Q_{r1} \geq Q_3$, and to $C(Q_2, Q_2) \backslash\{Q_M\}$ for Q_{r1}=Q_2. It is obvious to verify in table 1 of C that : for $Q_{r1} \geq Q_4$, [InfC(Q_{r1}, Q_{r1-1}), SupC(Q_{r1}, Q_{r1})] $\backslash\{Q_M\}$ = $[Q_{M-2}, Q_{M-1}]$; for $Q_2 \leq Q_{r1} \leq Q_3$, $C(Q_{r1}, Q_{r2})\backslash\{Q_M\}$ = $[Q_2, Q_{M-1}]$. Since for $Q_{r1} \geq Q_4$, $Q_\mu \in [Q_{M-2}, Q_{M-1}]$, then for $Q_2 \leq Q_{r1} \leq Q_3$ the interval $[Q_2, Q_{M-1}]$ can be restrained to the interval $[Q_{M-2}, Q_{M-1}]$. Therefor $Q_\mu \in [Q_{M-2}, Q_{M-1}]$.

Q_r	Q_α	Q_β	$Q_\lambda \in$
Q_2	Q_2	Q_2	$[Q_2, Q_3]$
Q_3	Q_2	Q_2	$[Q_3, Q_4]$
Q_3	Q_3	Q_2	$[Q_3, Q_5]\backslash\{Q_5\}$
Q_3	Q_3	Q_3	$[Q_3, Q_4]$
Q_4	Q_2	Q_2	$[Q_4, Q_5]$
Q_4	Q_3	Q_2	$[Q_4, Q_6]\backslash\{Q_6\}$
Q_4	Q_3	Q_3	$[Q_4, Q_5]$
Q_4	Q_4	Q_2	$[Q_5, Q_6]\backslash\{Q_6\}$
Q_4	Q_4	Q_3	$[Q_4, Q_6]\backslash\{Q_4, Q_6\}$
Q_4	Q_4	Q_4	$[Q_4, Q_5]\backslash\{Q_4\}$
Q_5	Q_2	Q_2	$[Q_5, Q_6]$
Q_5	Q_3	Q_2	$[Q_5, Q_6]$
Q_5	Q_3	Q_3	$[Q_5, Q_6]$
Q_5	Q_4	Q_3	$[Q_5, Q_6]\backslash\{Q_5\}$
Q_5	Q_4	Q_4	$[Q_5, Q6]\backslash\{Q_5\}$
Q_5	Q_5	Q_3	$\{Q_6\}$
Q_5	Q_5	Q_4	$[Q_5, Q_6]\backslash\{Q_5\}$
Q_5	Q_5	Q_5	$[Q_5, Q_6]\backslash\{Q_5\}$
Q_6	Q_2	Q_2	$\{Q_6\}$
Q_6	Q_3	Q_2	$\{Q_6\}$
Q_6	Q_3	Q_3	$\{Q_6\}$
Q_6	Q_4	Q_3	$\{Q_6\}$
Q_6	Q_4	Q_4	$\{Q_6\}$
Q_6	Q_5	Q_4	$\{Q_6\}$
Q_6	Q_5	Q_5	$\{Q_6\}$
Q_6	Q_6	Q_5	$\{Q_6\}$
Q_6	Q_6	Q_6	$\{Q_6\}$

Table 9