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On the checking of g-coherence of conditional probability bounds

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Abstract

We illustrate an approach to uncertain knowledge based on lower conditional probability bounds. Our results and algorithms exploit a concept of generalized coherence (g-coherence), which is a generalization of de Finetti's coherence principle and is equivalent to the "avoiding uniform loss" property for lower and upper probabilities(a la Walley). By our algorithms, given a g-coherent assessment, we can also correct it obtaining the associated coherent assessment (in the sense of Walley and Williams). Our algorithms work with a reduced set of variables and a reduced set of constraints. Such reduced sets are computed by suitably exploiting the additive structure of the random gains. In this paper, we study in detail imprecise assessments defined on families of three conditional events. We give some necessary and sufficient conditions and, then, we generalize some of the theoretical results obtained. We also exploit such results by proposing two algorithms which provide new strategies for reducing the number of constraints and for deciding g-coherence. Finally, we illustrate our approach by giving some examples.

Keywords: uncertain knowledge, probabilistic reasoning under coherence, lower conditional probability bounds, g-coherence checking, not relevant gains, basic sets, algorithms, computational aspects, reduced sets of variables, reduced sets of linear constraints.

1 Introduction

The probabilistic treatment of uncertainty by means of precise or imprecise probability assessments is well known. When the family of conditional events has no particular structure a suitable methodology is that based on de Finetti's coherence principle, or generalizations of it. It has been shown ([4]) that probabilistic reasoning under coherence is a proper generalization of system P, and that it is closely related to reasoning in probabilistic logic. We adopt the notion of coherence introduced in [11], renamed g-coherence (i.e. generalized coherence) in [1], and we can see that the notion given in [20] (see also [21]) is stronger than it. Actually, it can be shown that the notion of g-coherence is equivalent to the property of "avoiding uniform loss" given in [20]. As well known, probabilistic reasoning can be developed by local approaches based on local inference rules and global ones using linear optimization techniques (see, e.g., [17], [18]). The global incompleteness of the local approach has been extensively analyzed in [17], where it is shown that local probabilistic deduction is very limited in its field of application. The global approach to probabilistic reasoning has an exponential complexity. In particular, concerning probabilistic reasoning under coherence, in [3] it has been shown that the problems of deciding g-coherence and entailment (under g-coherence) are, respectively, NP-complete and co-NP-complete. Exploiting an idea given in [9] (see also [10]), an approach to the checking of coherence of conditional probability assessments allowing to split the problem into suitable sub-problems has been proposed in [7], [8]. An efficient procedure to propagate conditional probability bounds for families of *conjunctive* conditional events has been proposed in [16]. It can be shown ([3]) that the procedure proposed in [16] can be characterized in terms of random gains. In this paper we illustrate some results obtained in [5], [6]. The aim in such papers was that of diminishing the computational difficulties in the algorithms used for the checking of (generalized) coherence and propagation of imprecise conditional probability assessments ([1], [11], [12]). We describe an iterative procedure by means of which we can reduce the number of unknowns in the linear systems used in our algorithms. In particular, concerning the case of lower probability bounds, we illustrate an algorithm based on a mixed approach which allows to reduce the set of variables and/or the set of linear constraints. As remarked in [1], our algorithms can be also exploited for the checking of coherence (in the sense of Walley and Williams) and for producing (using the "least committal" correction connected with the principle of natural extension given in [20]) the coherent upper and lower probabilities. In this paper, we study in detail imprecise assessments defined on families of three conditional events. We make a minute analysis of many cases which can occur when the set of constituents is determined. We obtain some necessary and sufficient conditions and we generalize some theoretical results. Then, we exploit such results by proposing two algorithms which provide new strategies for reducing the number of constraints and for deciding g-coherence. Finally, we illustrate our approach by giving some examples. The remaining part of the paper is organized as follows. In Section 2 we give some preliminary concepts and results. Based on the notions of "not-relevant" gain and of "basic set", we show how to compute a reduced set of variables. We also describe an algorithm for reducing the number of linear constraints. In Section 3 we make a minute analysis of lower probability bounds defined on families of three conditional events. We obtain necessary and sufficient conditions for the g-coherence of the given imprecise assessment. In Section 4 we generalizes some of the theoretical results obtained in Section 3. In Section 5, we propose two algorithms which provide new strategies for reducing the number of constraints and for deciding g-coherence. In Section 6, we illustrate our methods by examining some examples. Finally, in Section 7 we give some conclusions and an outlook on further developments of this work.

2 Some preliminary concepts and results

For each integer n, we define $J_n = \{1, \ldots, n\}$. We denote by $\mathcal{F}_n$ a family of n conditional events $\{E_1|H_1, \ldots, E_n|H_n\}$ and by $\mathcal{A}_n$ a vector $(\alpha_1, \ldots, \alpha_n)$ of lower bounds on $\mathcal{F}_n$.

Definition 1 *The vector of probability lower bounds $\mathcal{A}_n$ is g-coherent iff there exists a coherent assessment $\{P(E_i|H_i), i \in J_n\}$ on $\mathcal{F}_n$ such that $P(E_i|H_i) \geq \alpha_i$.*

Remark 1 We observe that, for every $E|H$, such that $\emptyset \subset EH \subset H$, and for every $\alpha \in [0,1]$, the assessment $P(E|H) \geq \alpha$ is g-coherent.

Of course, in the definition above we can assume $\alpha_i > 0$, for each $i \in J_n$. More in general, given an arbitrary family of conditional events $\mathcal{F} = \{E_j|H_j\}_{j \in J}$ and a set of lower bounds $\mathcal{A}$ defined on $\mathcal{F}$, we have

Definition 2 *The set of probability lower bounds $\mathcal{A}$ defined on the family $\mathcal{F}$ is g-coherent iff, for every n and for every sub-family $\mathcal{F}_n \subseteq \mathcal{F}$, the vector of lower bounds $\mathcal{A}_n$ defined on $\mathcal{F}_n$ is g-coherent.*

We remark that, as shown by Definitions 1 and 2, the notion of g-coherence is based on that of coherence given for the case of precise assessments by de Finetti.
Given the pair $(\mathcal{F}_n, \mathcal{A}_n)$, we denote by $C_1, \ldots, C_m$ the constituents contained in $\mathcal{H}_n = \bigvee_{j \in J_n} H_j$. For each constituent $C_r, r \in J_m$, we introduce a vector $V_r = (v_{r1}, \ldots, v_{rn})$, where for each $i \in J_n$ it is respectively $v_{ri} = 1$, or $v_{ri} = 0$, or $v_{ri} = \alpha_i$, according to whether $C_r \subseteq E_i H_i$, or $C_r \subseteq E_i^c H_i$, or $C_r \subseteq H_i^c$. We denote by G_n the random gain $\sum_{j \in J_n} s_j H_j (E_j - \alpha_j), \;\; s_j \geq 0, \; j \in J_n$, and by

$$g_h = \sum_{j \in J_n} s_j (v_{hj} - \alpha_j) = \sum_{j : C_h \subseteq H_j} s_j (v_{hj} - \alpha_j) \tag{1}$$

the value of $G_n|\mathcal{H}_n$ associated with C_h. We denote by $(\mathcal{S}_n)$ the following system in the unknowns λ_r's.

$$(\mathcal{S}_n) \;\; \sum_{r=1}^{m} \lambda_r v_{ri} \geq \alpha_i, \;\; i \in J_n; \qquad \sum_{r=1}^{m} \lambda_r = 1; \qquad \lambda_r \geq 0, \;\; r \in J_m. \tag{2}$$

Then, as shown in [11], the set of lower bounds $\mathcal{A}$ defined on $\mathcal{F}$ is g-coherent iff, for every n and for every sub-family $\mathcal{F}_n \subseteq \mathcal{F}$, the system (2) is solvable. Moreover, based on a suitable alternative theorem, it can be shown ([1]) that the solvability of system (2) is equivalent to the following condition

$$Max \; G_n|\mathcal{H}_n \geq 0. \tag{3}$$

Then, exploiting the notion of random gain as usually made by other authors (see e.g. [20], [21]), the concept of g-coherence can be defined in the following alternative way.

Definition 3 *A set of lower bounds $\mathcal{A}$ defined on a family of conditional events $\mathcal{F}$ is g-coherent iff $\forall\ n, \forall\ \mathcal{F}_n \subseteq \mathcal{F}$ and $\forall\ s_j \geq 0, j \in J_n$, it is $Max\ G_n|\mathcal{H}_n \geq 0$.*

Now, we recall some definitions and theoretical conditions given in [2].

Definition 4 Let $\mathcal{G} = \{g_j\}_{j \in J_m}$ be the set of possible values of the random gain $G_n|\mathcal{H}_n$. Then, a value $g_r \in \mathcal{G}$ is said *"not relevant for the checking of condition (3)"*, or in short *"not relevant"*, if there exists a set $T_r \subseteq J_m \setminus \{r\}$ such that:

$$Max\ \{g_j\}_{j \in T_r}\ < 0 \implies g_r\ < 0\ . \tag{4}$$

Definition 5 A set $\mathcal{G}_\Gamma = \{g_r\}_{r \in \Gamma}$, with $\Gamma \subset J_m$, is said *not relevant* if, $\forall r \in \Gamma$, there exists a set $T_r \subseteq J_m \setminus \Gamma$ such that (4) is satisfied.

Given $r \in J_m$ and a set $\mathcal{T}_r \subseteq J_m \setminus \{r\}$, let us consider the following condition

$$g_r \leq \sum_{j \in \mathcal{T}_r} a_j g_j\ ; \quad a_j > 0\ ,\ \forall j \in \mathcal{T}_r\ ; \tag{5}$$

Based on Definition 4, it can be shown that, if for the gain g_r the above condition is satisfied, then g_r is not relevant. Exploiting in general the condition (5) we can reduce the number of variables. In particular, concerning the case of conjunctive events, we can improve the procedure given in [16]. We observe that $Max\ G_n|\mathcal{H}_n\ =\ Max\ \{g_j\}_{j \in J_m}$. Then, the following theorem illustrates the basic idea to reduce the number of unknowns ([2], [5]).

Theorem 1 *Let $\mathcal{T}$ be a strict subset of the set J_m such that for every $r \notin \mathcal{T}$ there exists $T_r \subseteq \mathcal{T}$ satisfying the condition (5). Then:*

$$Max\ \{g_j\}_{j \in J_m} \geq 0 \iff Max\ \{g_j\}_{j \in \mathcal{T}} \geq 0\ .$$

Based on the previous result and on suitable alternative theorems, the solvability of $(\mathcal{S}_n)$ is equivalent to the solvability of a system $(\mathcal{S}_n^{\mathcal{T}})$ which has a reduced number of unknowns. We denote respectively by $\Lambda_{\mathcal{T}} = (\lambda_r; r \in \mathcal{T})$ and $S_{\mathcal{T}}$ the vector of unknowns and the set of solutions of the system $(\mathcal{S}_n^{\mathcal{T}})$. Moreover, we define the linear function $\Phi_j^{\mathcal{T}}(\Lambda_{\mathcal{T}}) = \sum_{r: C_r \subseteq H_j} \lambda_r$. Then, we denote by $I_0^{\mathcal{T}}$ the (strict) subset of J_n defined as

$$I_0^{\mathcal{T}} = \{j \in J_n : M_j = Max_{\Lambda_{\mathcal{T}} \in S_{\mathcal{T}}} \Phi_j^{\mathcal{T}}(\Lambda_{\mathcal{T}}) = 0\}$$

and by $(\mathcal{F}_0^{\mathcal{T}}, \mathcal{A}_0^{\mathcal{T}})$ the pair associated with $I_0^{\mathcal{T}}$. The computation of a set $\mathcal{T}$ (called a *basic* set) is carried out by an iterative algorithm which checks the inequalities $g_r \leq g_h$, or $g_r \leq g_h + g_k$. Based on (1), we have the following relations

$$\begin{array}{rcl} g_r\ \leq\ g_h & \iff & V_r\ \leq\ V_h\ ; \\ g_r\ \leq\ g_h + g_k & \iff & \mathcal{A}_n\ \leq\ V_h + V_k - V_r\ . \end{array} \tag{6}$$

In the algorithm below the subsets of J_m are ordered and we denote by m_T (resp. M_T) the minimum (resp. maximum) of T and by $s(r)$ the element which follows r in the given ordering.

Algorithm 1 Let be given the triplet $(J_n, \mathcal{F}_n, \mathcal{A}_n)$.

1. Expand the expression $\bigwedge_{j \in J_n} \left(E_j H_j \vee E_j^c H_j \vee H_j^c\right)$. Then, determine the set $\{C_h\}_{h \in J_m}$ of constituents contained in $\mathcal{H}_n$ and the corresponding set of vectors $\{V_h\}_{h \in J_m}$.

2. For each subscript $r \in J_m$, eliminate the vector V_r if the condition $V_r \leq V_j$ is satisfied for a subscript $j \neq r$. Denote by $\{V_j\}_{j \in \mathcal{T}_0}$, where $\mathcal{T}_0 \subseteq J_m$, the set of remaining vectors.

3. Set $\mathcal{T}_1 = \mathcal{T}_1^c = \emptyset$ and $r = m_{\mathcal{T}_0}$.

4. If $r \in \mathcal{T}_1$, then go to Step 6. Otherwise, go to Step 5.

5. If a subset $\{h, k\} \subseteq \mathcal{T}_0 \setminus (\mathcal{T}_1^c \cup \{r\})$ is found such that the inequality $\mathcal{A}_n \leq V_h + V_k - V_r$ holds, then replace respectively $\mathcal{T}_1^c$ by $\mathcal{T}_1^c \cup \{r\}$ and $\mathcal{T}_1$ by $\mathcal{T}_1 \cup \{h, k\}$. Otherwise, replace $\mathcal{T}_1$ by $\mathcal{T}_1 \cup \{r\}$.

6. If $r = M_{\mathcal{T}_0}$, go to Step 7. Otherwise, replace r by $s(r)$ and go to Step 4.

7. If $\mathcal{T}_1 \subset \mathcal{T}_0$, then introduce two sets $\mathcal{T}_2, \mathcal{T}_2^c$ and replace $(\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_1^c)$ by $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_2^c)$. Then, go to Step 3. Otherwise, $\mathcal{T} = \mathcal{T}_1$ and the procedure ends.

The algorithm stops when, for some i, one has $\mathcal{T}_{i-1} = \mathcal{T}_i$. Then, the output $\mathcal{T} = \mathcal{T}_i$ is obtained.

In [2] the following theorem has been proved.

Theorem 2 *The imprecise assessment $\mathcal{A}_n$ on $\mathcal{F}_n$ is g-coherent if and only if the following conditions are satisfied:*

1. the system $(\mathcal{S}_n^{\mathcal{T}})$ is solvable;

2. if $I_0^{\mathcal{T}} \neq \emptyset$, then $\mathcal{A}_0^{\mathcal{T}}$ is g-coherent.

We now describe a procedure which allows to reduce the family $\mathcal{F}_n$ on which to construct the system $(\mathcal{S}_n^{\mathcal{T}})$.

Given the set $\mathcal{V} = \{V_1, \ldots, V_m\}$, we define

$$\mathcal{W} = \{V_r \in \mathcal{V} : v_{ri} \neq 0, \forall\, i \in J_n\} \tag{7}$$

and, for each $V_r \in \mathcal{W}$, the set

$$N_r = \{i \in J_n : C_r \subseteq H_i^c\} \; . \tag{8}$$

Then, we define

$$\mathcal{V}_k = \{V_r \in \mathcal{W} : |N_r| = k\}, \; k = 0, 1, \ldots, n-1 \; . \tag{9}$$

One has

Theorem 3 *Assume that $\mathcal{W}$ is not empty. Then:*
1. the system $(\mathcal{S}_n)$ is solvable;
2. given any $V_h \in \mathcal{W}$, if the sub-vector $\mathcal{A}_{N_h}$ associated with N_h is g-coherent, then $\mathcal{A}_n$ is g-coherent.

Theorem 4 *Assume that $\mathcal{W}$ is not empty and let V_h, V_k two vectors in $\mathcal{W}$. If the sub-vector $\mathcal{A}_{N_h \cap N_k}$ associated with $N_h \cap N_k$ is g-coherent, then $\mathcal{A}_n$ is g-coherent.*

We observe that, if $\mathcal{A}_n$ is g-coherent, then for each $N_h \subset J_n$ the assessment $\mathcal{A}_{N_h}$ on $\mathcal{F}_{N_h}$ is g-coherent. Then, Theorems 3 and 4 can be re-formulated in the following more expressive way.

Theorem 5 *Assume that $\mathcal{W}$ is not empty. Then:*
1. the system $(\mathcal{S}_n)$ is solvable;
2. $\mathcal{A}_n$ is g-coherent iff, for each given $V_h \in \mathcal{W}$, the sub-vector $\mathcal{A}_{N_h}$ associated with N_h is g-coherent.

Theorem 6 *Assume that $|\mathcal{W}| \geq 2$. Then:*
1. the system $(\mathcal{S}_n)$ is solvable;
2. $\mathcal{A}_n$ is g-coherent iff, for each pair of vectors V_h, V_k in $\mathcal{W}$, the sub-vector $\mathcal{A}_{N_h \cap N_k}$ associated with $N_h \cap N_k$ is g-coherent.

Theorem 6 can be generalized, in an obvious way, to any given subset $\{V_{h_1}, \ldots, V_{h_t}\} \subseteq \mathcal{W}$.

Given as input the pair $(\mathcal{F}, \mathcal{A}) = (\mathcal{F}_n, \mathcal{A}_n)$ the following procedure, called $SubF\mathcal{V}()$, returns a suitable pair $(\mathcal{F}, \mathcal{A}) = SubF\mathcal{V}(\mathcal{F}_n, \mathcal{A}_n)$.

Algorithm 2 $SubF\mathcal{V}$ $(\mathcal{F}, \mathcal{A})$

1. Set $k = 0$.
2. Determine $\mathcal{V}_k$.
3. If $\mathcal{V}_k \neq \emptyset$ and $k = 0$, return $(\emptyset, \emptyset)$.
   If $\mathcal{V}_k = \emptyset$ and $k < n - 1$, then set $k = k + 1$ and go to Step 2.
   If $\mathcal{V}_k \neq \emptyset$, given any $V_r \in \mathcal{V}_k$, let $(\mathcal{F}_{N_r}, \mathcal{A}_{N_r})$ be the pair associated with N_r. Then, set $(\mathcal{F}, \mathcal{A}) = (\mathcal{F}_{N_r}, \mathcal{A}_{N_r})$ and go to Step 1.
   If $\mathcal{V}_k = \emptyset$ and $k = n - 1$, return $(\mathcal{F}, \mathcal{A})$.

3 Checking of g-coherence: the case $n = 3$

In this section, with the aim of better illustrating the previous results, we will examine with great details the case $n = 3$. Let $\mathcal{A} = (\alpha_1, \alpha_2, \alpha_3)$ be a vector of lower bounds on a family $\mathcal{F} = \{E_1|H_1, E_2|H_2, E_3|H_3\}$. To avoid the analysis of trivial cases, we assume

$$\emptyset \subset E_i H_i \subset H_i \ , \quad i = 1, 2, 3.$$

Remark 2 In what follows, as it will be specified, to avoid the analysis of particular cases, we will also assume

$$\alpha_i < 1 \ , \quad i = 1, 2, 3.$$

We observe that, as $\alpha_i < 1$, given a given vector V_r, if $v_{ri} = 1$, then $C_r \subseteq E_i H_i$.

As $n = 3$, the set of vectors $\mathcal{V} = \{V_1, \ldots, V_m\}$, where $m \leq 26$, is a subset of the set

$$\{(1,1,1), (1,1,\alpha_3), (1,\alpha_2,1), (\alpha_1,1,1), \ldots, (\alpha_1,0,0), (0,\alpha_2,0), (0,0,\alpha_3), (0,0,0)\} \ .$$

We represent $\mathcal{V}$ in the following way

$$\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_4 \cup \mathcal{U}_5 \cup \mathcal{U}_6 \ ,$$

where

$$\mathcal{V}_0 \subseteq \{(1,1,1)\}\ , \quad \mathcal{V}_1 \subseteq \{(1,1,\alpha_3),(1,\alpha_2,1),(\alpha_1,1,1)\}\ ,$$

$$\mathcal{V}_2 \subseteq \{(1,\alpha_2,\alpha_3),(\alpha_1,1,\alpha_3),(\alpha_1,\alpha_2,1)\}\ , \quad \mathcal{U}_1 \subseteq \{(1,1,0),(1,0,1),(0,1,1)\}\ ,$$

$$\mathcal{U}_2 \subseteq \{(1,\alpha_2,0),(1,0,\alpha_3),(\alpha_1,1,0),(0,1,\alpha_3),(\alpha_1,0,1),(0,\alpha_2,1),\}\ ,$$

$$\mathcal{U}_3 \subseteq \{(1,0,0),(0,1,0),(0,0,1)\}\ , \quad \mathcal{U}_4 \subseteq \{(\alpha_1,\alpha_2,0),(\alpha_1,0,\alpha_3),(0,\alpha_2,\alpha_3)\}\ ,$$

$$\mathcal{U}_5 \subseteq \{(\alpha_1,0,0),(0,\alpha_2,0),(0,0,\alpha_3)\}\ , \quad \mathcal{U}_6 \subseteq \{(0,0,0)\}\ .$$

Based on Theorem 3, we have the following results.

Theorem 7 *If $|\mathcal{V}_0| = 1$, then $\mathcal{A}$ is g-coherent.*

Proof. In this case $\mathcal{V}_0 = \{V_1\}$, where $V_1 = (1,1,1)$. Then, by Theorem 3, $(\mathcal{S}_3)$ is solvable and $N_1 = \emptyset$, and hence $\mathcal{A}$ is g-coherent.

Theorem 8 *If $\mathcal{V}_0 = \emptyset$ and $|\mathcal{V}_1| \geq 1$, then $\mathcal{A}$ is g-coherent.*

Proof. In this case, there exists a vector V_1, with $V_1 = (1,1,\alpha_3)$, or $V_1 = (1,\alpha_2,1)$, or $V_1 = (\alpha_1,1,1)$, such that $V_1 \in \mathcal{V}_1$. Then, by Theorem 3, $(\mathcal{S}_3)$ is solvable. Moreover, $|N_1| = 1$ and, of course, the assessment $\mathcal{A}_{N_1}$ on $\mathcal{F}_{N_1}$ is g-coherent, hence $\mathcal{A}$ is g-coherent.

Theorem 9 *If $\mathcal{V}_0 = \mathcal{V}_1 = \emptyset$ and $|\mathcal{V}_2| \geq 2$, then $\mathcal{A}$ is g-coherent.*

Proof. In this case, there exist two vectors V_1, V_2, with

$$V_1 = (1,\alpha_2,\alpha_3)\ , \quad V_2 = (\alpha_1,1,\alpha_3)\ ,$$

or

$$V_1 = (1,\alpha_2,\alpha_3)\ , \quad V_2 = (\alpha_1,\alpha_2,1)\ ,$$

or

$$V_1 = (\alpha_1,1,\alpha_3)\ , \quad V_2 = (\alpha_1,\alpha_2,1)\ ,$$

such that $\{V_1, V_2\} \subseteq \mathcal{V}_2$. Then, $|N_1| = |N_2| = 2$ and $N_1 \cap N_2 = 1$, so that the assessment $\mathcal{A}_{N_1 \cap N_2}$ on $\mathcal{F}_{N_1 \cap N_2}$ is g-coherent. Hence, by Theorem 4, $\mathcal{A}$ is g-coherent

Theorem 10 *If $\mathcal{V}_0 = \mathcal{V}_1 = \emptyset$, $|\mathcal{V}_2| = 1$, with $\mathcal{V}_2 = \{(1,\alpha_2,\alpha_3)\}$, and the quasi-conjunction of the conditional events in the family $\mathcal{F}_2 = \{E_2|H_2, E_3|H_3\}$ is verifiable, then $\mathcal{A}$ is g-coherent. The same conclusion holds if $\mathcal{V}_2 = (\alpha_1,1,\alpha_3)$ and $\mathcal{F}_2 = \{E_1|H_1, E_3|H_3\}$, or $\mathcal{V}_2 = (\alpha_1,\alpha_2,1)$ and $\mathcal{F}_2 = \{E_1|H_1,\ E_2|H_2\}$.*

Proof. We only examine the case $\mathcal{V}_2 = \{V_1\} = \{(1,\alpha_2,\alpha_3)\}$ because the reasoning is similar in the other cases. It is $N_1 = \{2,3\}$. Then, by Theorem 3, $(\mathcal{S}_3)$ is solvable, so that g-coherence of $\mathcal{A}$ amounts to g-coherence of the assessment $\mathcal{A}_2 = (\alpha_2,\alpha_3)$ on $\mathcal{F}_2 = \{E_2|H_2, E_3|H_3\}$. The quasi-conjunction of $E_2|H_2, E_3|H_3$ is the conditional event $(E_2H_2 \vee H_2^c) \wedge (E_3H_3 \vee H_3^c)\,|\,(H_2 \vee H_3)$, which is verifiable iff $E_2H_2E_3H_3 \vee E_2H_2H_3^c \vee H_2^cE_3H_3 \neq \emptyset$. Therefore, denoting by $\mathcal{U}$ the intersection of the set of vectors relative to the pair $(\mathcal{F}_2, \mathcal{A}_2)$ with the set $\{(1,1),(1,\alpha_3),(\alpha_2,1)\}$, one has $\mathcal{U} \neq \emptyset$. Then, we can apply Theorem 7 or Theorem 8 to the pair $(\mathcal{F}_2, \mathcal{A}_2)$ obtaining that $\mathcal{A}_2$ is g-coherent, so that $\mathcal{A}$ is g-coherent too.

Remark 3 We observe that, concerning the previous result, if the quasi conjunction of $E_2|H_2, E_3|H_3$ is not verifiable, then g-coherence of $\mathcal{A}$ requires that some numerical conditions be satisfied by $\alpha_1, \alpha_2, \alpha_3$.

We have

Theorem 11 *If $\mathcal{V}_0 = \mathcal{V}_1 = \emptyset$, $\mathcal{V}_2 = \{(1, \alpha_2, \alpha_3)\}$ and the quasi-conjunction of the conditional events in the family $\mathcal{F}_2 = \{E_2|H_2, E_3|H_3\}$ is not verifiable, then $\mathcal{A}$ is g-coherent iff $\alpha_2 + \alpha_3 \leq 1$.*

Proof. Let us define $V_1 = (1, \alpha_2, \alpha_3)$. Then, by Theorem 3, $(\mathcal{S}_3)$ is solvable and $N_1 = \{2, 3\}$, so that we need to examine the pair $(\mathcal{F}_2, \mathcal{A}_2)$. By the hypotheses it follows that the set of vectors relative to $(\mathcal{F}_2, \mathcal{A}_2)$ contains the set $\{(1, 0), (0, 1)\}$. Then, it is easy to verify that $\mathcal{A}_2$ (and hence $\mathcal{A}$) is g-coherent iff $\alpha_2 + \alpha_3 \leq 1$.

Remark 4 We observe that, under similar hypotheses, Theorem 11 can also be proved, obtaining similar results, in the following cases

1. $\mathcal{V}_2 = \{(\alpha_1, 1, \alpha_3)\}$, $\mathcal{F}_2 = \{E_1|H_1, E_3|H_3\}$;

2. $\mathcal{V}_2 = \{(\alpha_1, \alpha_2, 1)\}$, $\mathcal{F}_2 = \{E_1|H_1, E_2|H_2\}$.

We remark that, if $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \emptyset$, then we need to check the solvability of the system $(\mathcal{S}_3)$, or of the system $(\mathcal{S}_3^{\mathcal{T}})$ exploiting a basic set $\mathcal{T}$. Some results are given below.

Theorem 12 *If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \emptyset$ and $\alpha_1 + \alpha_2 + \alpha_3 > 2$, then $\mathcal{A}$ is not g-coherent.*

Proof. We observe that

$$\mathcal{V} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_4 \cup \mathcal{U}_5 \cup \mathcal{U}_6 \ .$$

Moreover, the system $(\mathcal{S}_3)$ is solvable iff there exists a vector $\Lambda = (\lambda_1, \ldots, \lambda_m)$ such that

$$\sum_{h=1}^{m} \lambda_h V_h \geq \mathcal{A} \ , \quad \sum_{h=1}^{m} \lambda_h = 1 \ , \quad \lambda_h \geq 0 \ , \ \forall \, h \ .$$

Then, defining

$$V_1^* = (1, 1, 0) \ , \quad V_2^* = (1, 0, 1) \ , \quad V_3^* = (0, 1, 1) \ ,$$

for each $\mathcal{V}_r \in \mathcal{V}$ it is $V_r \leq V_1^*$, or $V_r \leq V_2^*$, or $V_r \leq V_3^*$. Then, based on any solution $(\lambda_1, \ldots, \lambda_m)$ of $(\mathcal{S}_3)$ we would obtain

$$V^* = (v_1^*, v_2^*, v_3^*) = \lambda_1^* V_1^* + \lambda_2^* V_2^* + \lambda_3^* V_3^* \geq \mathcal{A} \ , \tag{10}$$

where

$$\lambda_1^* = \sum_{r : V_r \leq V_1^*} \lambda_r \ , \quad \lambda_2^* = \sum_{r : V_r \nleq V_1^*, V_r \leq V_2^*,} \lambda_r \ , \quad \lambda_3^* = \sum_{r : V_r \nleq V_1^*, V_r \nleq V_2^*, V_r \leq V_3^*} \lambda_r \ ,$$

and with $v_1^* + v_2^* + v_3^* = 2$. The proof follows by observing that, if the condition (10) were satisfied, then it would be $\alpha_1 + \alpha_2 + \alpha_3 \leq 2$, which is a contradiction.

Theorem 13 *If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \emptyset$, $|\mathcal{U}_1| = 3$, $\alpha_i < 1 \, \forall \, i$ then one has:*
a) there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = 3$;
b) $\mathcal{A}$ is g-coherent iff $\alpha_1 + \alpha_2 + \alpha_3 \leq 2$.

Proof. a) We represent $\mathcal{U}_1$ as the set $\{V_1, V_2, V_3\}$, with

$$V_1 = (0,1,1)\ ,\ \ V_2 = (1,0,1)\ ,\ \ V_3 = (1,1,0)\ .$$

For each $r > 3$ there exists $h \in \{1,2,3\}$ such that $V_r \leq V_h$ and hence g_r is not relevant. Then, $\mathcal{T} = \{1,2,3\}$ is a basic set.
b.1) If $\alpha_1 + \alpha_2 + \alpha_3 > 2$, then by Theorem 12 it follows that $\mathcal{A}$ is not g-coherent.
b.2) Let us assume that $\alpha_1 + \alpha_2 + \alpha_3 \leq 2$. If $\alpha_1 + \alpha_2 \leq 1$, then

$$V^* = (\alpha_1, 1-\alpha_1, 1)\ \geq\ (\alpha_1, \alpha_2, 1)\ \geq\ \mathcal{A}\ ,$$

and

$$V^*\ =\ (1-\alpha_1)V_1 + \alpha_1 V_2\ .$$

Therefore, the vector $(\lambda_1, \lambda_2, \lambda_3) = (1-\alpha_1, \alpha_1, 0)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$. Moreover, it is $I_0^{\mathcal{T}} = \emptyset$, and hence $\mathcal{A}$ is g-coherent.
If $\alpha_1 + \alpha_2 > 1$, then the segment (α_1, α_2, z), $0 \leq z \leq 1$, intersects the triangle $V_1V_2V_3$ in the point $V^* = (\alpha_1, \alpha_2, 2-\alpha_1-\alpha_2) \geq \mathcal{A}$. Moreover,

$$V^*\ =\ \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3\ ,$$

with

$$\lambda_1 = 1-\alpha_1\ ,\quad \lambda_2 = 1-\alpha_2\ ,\quad \lambda_3 = \alpha_1 + \alpha_2 - 1\ .$$

Then, the vector $(\lambda_1, \lambda_2, \lambda_3)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence $\mathcal{A}$ is g-coherent.

Theorem 14 *If* $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \emptyset$, $\mathcal{U}_1 = \{V_1, V_2\} = \{(1,1,0), (1,0,1)\}$, $\{V_3, V_4\} = \{(0,1,\alpha_3), (0,\alpha_2,1)\} \subseteq \mathcal{U}_2$, $\alpha_i < 1\,\forall\, i$, *then one has:*
a) for every $r > 4$, *the gain* g_r *is not relevant;*
b) if $\alpha_1 + \alpha_2 \leq 1$, *or* $\alpha_2 + \alpha_3 \leq 1$, *or* $\alpha_1 + \alpha_3 \leq 1$, *then there exists a basic set* $\mathcal{T}$, *with* $|\mathcal{T}| \leq 3$, *and* $\mathcal{A}$ *is g-coherent;*
c) if $\alpha_1 + \alpha_2 > 1, \alpha_2 + \alpha_3 > 1, \alpha_1 + \alpha_3 > 1$, *then there exists a basic set* $\mathcal{T}$, *with* $|\mathcal{T}| \leq 3$, *and* $\mathcal{A}$ *is g-coherent iff*

$$\alpha_1\alpha_2 + \alpha_3 \leq 1 \quad \textit{or} \quad \alpha_1\alpha_3 + \alpha_2 \leq 1.$$

Proof. a) By the hypotheses, it follows that for each $V_r \in \mathcal{V}$, with $r > 4$, there exists $h \in \{1,2,3,4\}$ such that $V_r \leq V_h$, therefore g_r is not relevant.
We observe that the gains associated with the vectors V_1, V_2, V_3, V_4 are respectively

$$\begin{aligned}
g_1 &= s_1(1-\alpha_1) + s_2(1-\alpha_2) - s_3\alpha_3\ ,\\
g_2 &= s_1(1-\alpha_1) - s_2\alpha_2 + s_3(1-\alpha_3)\ ,\\
g_3 &= -s_1\alpha_1 + s_2(1-\alpha_2)\ ,\\
g_4 &= -s_1\alpha_1 + s_3(1-\alpha_3)\ .
\end{aligned}$$

In order to study the g-coherence of $\mathcal{A}$ we need to consider the equations of the planes containing the triangles $V_1V_2V_3$, $V_1V_2V_4$, $V_1V_3V_4$ and $V_2V_3V_4$. These equations are, respectively,

$$\alpha_3 x + y + z = 1 + \alpha_3\ ,$$

$$\alpha_2 x + y + z = 1 + \alpha_2\ ,$$

$$\alpha_3(1-\alpha_2)x + (1-\alpha_3)y + (1-\alpha_2)z = 1 - \alpha_2\alpha_3\ ,$$

$$\alpha_2(1-\alpha_3)x + (1-\alpha_3)y + (1-\alpha_2)z = 1 - \alpha_2\alpha_3\ .$$

The intersection points of the segment (x, α_2, α_3), $0 \leq x \leq 1$, with the planes containing the triangles $V_1V_2V_3$ and $V_1V_2V_4$, are respectively $V_x^* = (\frac{1-\alpha_2}{\alpha_3}, \alpha_2, \alpha_3)$ and $V_x^{**} = (\frac{1-\alpha_3}{\alpha_2}, \alpha_2, \alpha_3)$, with

$$V_x^* \geq \mathcal{A} \quad \Longleftrightarrow \quad \alpha_1\alpha_3 + \alpha_2 \leq 1 \; ;$$

$$V_x^{**} \geq \mathcal{A} \quad \Longleftrightarrow \quad \alpha_1\alpha_2 + \alpha_3 \leq 1 \; .$$

The intersection point of the segment (α_1, y, α_2), $0 \leq y \leq 1$, with the plane containing the triangle $V_1V_3V_4$ is $V_y^* = (\alpha_1,\, 1 - \alpha_3 - \alpha_1\alpha_3 + \alpha_1\alpha_2\alpha_3,\, \alpha_3)$, with

$$V_y^* \geq \mathcal{A} \quad \Longleftrightarrow \quad \alpha_1\alpha_3 + \alpha_3 \leq 1 \; .$$

The intersection point of the segment (α_1, α_2, z), $0 \leq z \leq 1$, with the plane containing the triangle $V_2V_3V_4$ is

$$V_z^* = (\alpha_1, \alpha_2, \tfrac{1-\alpha_2-\alpha_1\alpha_2(1-\alpha_3)}{1-\alpha_2}) \geq \mathcal{A} \; , \qquad \forall \; \alpha_3 \in [0, 1] \; .$$

b.1) If $\alpha_1 + \alpha_2 \leq 1$, then

$$\forall \, \lambda \in (\frac{\alpha_2}{1-\alpha_2}, \frac{1-\alpha_1}{\alpha_1}) \; , \quad \forall \, a \; \geq Max \, \{\frac{1-\alpha_1}{1-(1+\lambda)\alpha_1}, \frac{1-\alpha_2}{\lambda-(1+\lambda)\alpha_2}\} \; ,$$

one has

$$ag_2 + \lambda a g_3 \; \geq \; g_1 \; .$$

Therefore g_1 is not relevant and $\mathcal{T} = \{2, 3, 4\}$ is a basic set. Moreover,

$$V_z^* = \lambda_2 V_2 + \lambda_3 V_3 + \lambda_4 V_4 \; ,$$

with

$$\lambda_2 = \alpha_1 \; , \quad \lambda_3 = \frac{\alpha_1\alpha_2}{1-\alpha_2} \; , \quad \lambda_4 = \frac{1-\alpha_1-\alpha_2}{1-\alpha_2} \; .$$

We recall that $\alpha_i > 0, i = 1, 2, 3$, so that $\lambda_i \geq 0, i = 2, 3, 4$. Then, the vector $(\lambda_2, \lambda_3, \lambda_4)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $|I_0^{\mathcal{T}}| \leq 1$, and hence $\mathcal{A}$ is g-coherent.

b.2) If $\alpha_2 + \alpha_3 \leq 1$, then there exist suitable positive quantities a, λ, b, δ such that

$$ag_1 + \lambda a g_2 \; \geq \; g_3 \; , \quad bg_1 + \delta b g_2 \; \geq \; g_4 \; ,$$

therefore g_3 and g_4 are not relevant and $\mathcal{T} = \{1, 2\}$ is a basic set. Moreover, $V^* = (1, \alpha_2, 1 - \alpha_2) \geq (1, \alpha_2, \alpha_3) \geq \mathcal{A}$ and one has

$$V^* = \lambda_1 V_1 + \lambda_2 V_2 \; ,$$

with

$$\lambda_1 = \alpha_2 \; , \quad \lambda_2 = 1 - \alpha_2 \; .$$

Then, the vector (λ_1, λ_2) is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence $\mathcal{A}$ is g-coherent.

b.3) If $\alpha_1 + \alpha_3 \leq 1$, then $\alpha_1\alpha_3 + \alpha_3 \leq 1$ and hence $V_y^* \geq \mathcal{A}$. Moreover, there exist suitable positive quantities a, λ such that

$$ag_1 + \lambda a g_4 \; \geq \; g_2 \; ,$$

therefore g_2 is not relevant and $\mathcal{T} = \{1, 3, 4\}$ is a basic set. We have

$$V_y^* = \lambda_1 V_1 + \lambda_3 V_3 + \lambda_4 V_4 \; ,$$

with

$$\lambda_1 = \alpha_1 \ , \quad \lambda_3 = \frac{1-(\alpha_1+\alpha_3)}{1-\alpha_3} \ , \quad \lambda_4 = \frac{\alpha_1\alpha_3}{1-\alpha_3} \ .$$

Then, the vector $(\lambda_1, \lambda_3, \lambda_4)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $|I_0^{\mathcal{T}}| \leq 1$, and hence $\mathcal{A}$ is g-coherent.
Therefore, under the condition

$$\alpha_1 + \alpha_2 \leq 1 \ , \quad \text{or} \quad \alpha_2 + \alpha_3 \leq 1 \ , \quad \text{or} \quad \alpha_1 + \alpha_3 \leq 1 \ ,$$

$\mathcal{A}$ is g-coherent.
c) Let us assume that $\alpha_1 + \alpha_2 > 1, \ \alpha_2 + \alpha_3 > 1, \ \alpha_1 + \alpha_3 > 1$.
c.1) If $\alpha_1\alpha_3 + \alpha_2 \leq 1$, then $V_x^* \geq \mathcal{A}$. Moreover, there exist suitable positive quantities a, λ such that

$$ag_2 + \lambda a g_3 \ \geq \ g_4 \ ,$$

therefore g_4 is not relevant and $\mathcal{T} = \{1,2,3\}$ is a basic set. We have $V_x^* = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3$, with

$$\lambda_1 = \frac{(1-\alpha_2)(1-\alpha_3)}{\alpha_3} \ , \quad \lambda_2 = 1-\alpha_2 \ , \quad \lambda_3 = \frac{\alpha_2+\alpha_3-1}{\alpha_3} \ .$$

Then, the vector $(\lambda_1, \lambda_2, \lambda_3)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence $\mathcal{A}$ is g-coherent.
c.2) If $\alpha_1\alpha_2 + \alpha_3 \leq 1$, then $V_x^{**} \geq \mathcal{A}$. Moreover, there exist suitable positive quantities a, λ such that

$$ag_1 + \lambda a g_4 \ \geq \ g_3 \ ,$$

therefore g_3 is not relevant and $\mathcal{T} = \{1,2,4\}$ is a basic set. We have $V_x^{**} = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_4 V_4$, with

$$\lambda_1 = 1-\alpha_3 \ , \quad \lambda_2 = \frac{(1-\alpha_2)(1-\alpha_3)}{\alpha_2} \ , \quad \lambda_4 = \frac{\alpha_2+\alpha_3-1}{\alpha_2} \ .$$

Then, the vector $(\lambda_1, \lambda_2, \lambda_4)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence $\mathcal{A}$ is g-coherent.
3. If $\alpha_1\alpha_2 + \alpha_3 > 1$ and $\alpha_1\alpha_3 + \alpha_2 > 1$, then the condition

$$\lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3 + \lambda_4 V_4 \geq \mathcal{A}$$

is not satisfiable, that is $(\mathcal{S}_3^{\mathcal{T}})$ is not solvable.
In conclusion, under the hypothesis $\alpha_1 + \alpha_2 > 1, \ \alpha_2 + \alpha_3 > 1, \ \alpha_1 + \alpha_3 > 1$, $\mathcal{A}$ is g-coherent iff $\alpha_1\alpha_2 + \alpha_3 \leq 1$ or $\alpha_1\alpha_3 + \alpha_2 \leq 1$.

Remark 5 We observe that, under similar hypotheses, Theorem 14 can also be proved, obtaining similar results, in the following cases

$$\mathcal{U}_1 = \{V_1, V_2\} = \{(1,1,0),(0,1,1)\}, \ \{V_3, V_4\} = \{(1,0,\alpha_3),(\alpha_1,0,1)\} \subseteq \mathcal{U}_2$$

$$\mathcal{U}_1 = \{V_1, V_2\} = \{(1,0,1),(0,1,1)\}, \ \{V_3, V_4\} = \{(1,\alpha_2,0),(\alpha_1,1,0)\} \subseteq \mathcal{U}_2.$$

Theorem 15 *If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \emptyset$, $\mathcal{U}_1 = \{V_1, V_2\} = \{(1,1,0),(1,0,1)\}, V_3 = (0,1,\alpha_3) \in \mathcal{U}_2, (0,\alpha_2,1) \notin \mathcal{U}_2, \ \alpha_i < 1 \, \forall \, i$, then one has:*
a) for every $r > 3$, the gain g_r is not relevant;
b) if $\alpha_2 + \alpha_3 \leq 1$, then there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = 2$, and $\mathcal{A}$ is g-coherent;
c) if $\alpha_2 + \alpha_3 > 1$, then $\mathcal{A}$ is g-coherent iff $\alpha_1\alpha_3 + \alpha_2 \leq 1$.

Proof. a) By the hypotheses, it follows that for each $V_r \in \mathcal{V}$, with $r > 3$, there exists $h \in \{1,2,3\}$ such that $V_r \leq V_h$, therefore g_r is not relevant.
b) If $\alpha_2 + \alpha_3 \leq 1$, then

$$V^* = (1, \alpha_2, 1-\alpha_2) \geq (1, \alpha_2, \alpha_3) \geq \mathcal{A}\,.$$

Moreover, there exist suitable positive quantities a, λ such that

$$ag_1 + \lambda a g_2 \;\geq\; g_3\,,$$

therefore g_3 is not relevant and $\mathcal{T} = \{1,2\}$ is a basic set. We have $V^* = \alpha_2 V_1 + (1-\alpha_2)V_2$. Then, the vector $(\alpha_2, 1-\alpha_2)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence $\mathcal{A}$ is g-coherent.
c) Let us assume that $\alpha_2 + \alpha_3 > 1$, $\alpha_1\alpha_3 + \alpha_2 \leq 1$. We recall that the equation of the plane containing the triangle $V_1V_2V_3$ is $\alpha_3 x + y + z = 1 + \alpha_3$. Then, the segment $(x, \alpha_2, \alpha_3), 0 \leq x \leq 1$, intersects the triangle $V_1V_2V_3$ in the point $V_x^* = (\frac{1-\alpha_2}{\alpha_3}, \alpha_2, \alpha_3)$ and one has $V_x^* \geq \mathcal{A}$. Moreover,

$$V_x^* = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3,$$

with

$$\lambda_1 = \frac{(1-\alpha_1)(1-\alpha_3)}{\alpha_3}\;, \quad \lambda_2 = 1-\alpha_2\;, \quad \lambda_3 = \frac{\alpha_2+\alpha_3-1}{\alpha_3}\;.$$

Then, the vector $(\lambda_1, \lambda_2, \lambda_3)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence $\mathcal{A}$ is g-coherent.
We observe that, if $\alpha_1\alpha_3 + \alpha_2 > 1$, then the condition

$$\lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3 \geq \mathcal{A}$$

is not satisfiable, that is $(\mathcal{S}_3^{\mathcal{T}})$ is not solvable. Therefore, under the hypothesis $\alpha_2 + \alpha_3 > 1$, $\mathcal{A}$ is g-coherent iff $\alpha_1\alpha_3 + \alpha_2 \leq 1$.

Remark 6 We observe that, under similar hypotheses, Theorem 15 can also be proved, obtaining similar results, in the following cases

$$\mathcal{U}_1 = \{V_1, V_2\} = \{(1,1,0),(1,0,1)\},\; V_3 = (0,\alpha_2,1) \in \mathcal{U}_2,\; V_4 = (0,1,\alpha_3) \notin \mathcal{U}_2$$

$$\mathcal{U}_1 = \{V_1, V_2\} = \{(1,1,0),(0,1,1)\},\; V_3 = (1,0,\alpha_3) \in \mathcal{U}_2,\; V_4 = (\alpha_1,0,1) \notin \mathcal{U}_2$$

$$\mathcal{U}_1 = \{V_1, V_2\} = \{(1,1,0),(0,1,1)\},\; V_3 = (\alpha_1,0,1) \in \mathcal{U}_2,\; V_4 = (1,0,\alpha_3) \notin \mathcal{U}_2$$

$$\mathcal{U}_1 = \{V_1, V_2\} = \{(1,0,1),(0,1,1)\},\; V_3 = (1,\alpha_2,0) \in \mathcal{U}_2,\; V_4 = (\alpha_1,1,0) \notin \mathcal{U}_2$$

$$\mathcal{U}_1 = \{V_1, V_2\} = \{(1,0,1),(0,1,1)\},\; V_3 = (1,\alpha_2,0) \in \mathcal{U}_2,\; V_4 = (\alpha_1,1,0) \notin \mathcal{U}_2.$$

Theorem 16 *If* $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \emptyset$, $\mathcal{U}_1 = \{V_1\} = \{(1,1,0)\}, \{V_2, V_3, V_4, V_5\} = \{(\alpha_1,0,1),(0,\alpha_2,1),(1,0,\alpha_3),(0,1,\alpha_3)\} \subseteq \mathcal{U}_2,\; \alpha_i < 1\,\forall\, i$, *then one has:*
a) for every $r > 5$, *the gain* g_r *is not relevant;*
b) if $\alpha_1 + \alpha_2 \leq 1$, *then there exists a basic set* $\mathcal{T}$, *with* $|\mathcal{T}| = 3$. *Moreover,* $\mathcal{A}$ *is g-coherent.*
c) if $\alpha_1 + \alpha_2 > 1$, *then there exists a basic set* $\mathcal{T}$, *with* $|\mathcal{T}| = 3$, *and* $\mathcal{A}$ *is g-coherent iff*

$$\alpha_3 \leq Max\,\{\frac{\alpha_1+\alpha_2-2\alpha_1\alpha_2}{\alpha_1+\alpha_2-\alpha_1\alpha_2}\;,\; 1-\alpha_1+\alpha_1\alpha_3-\alpha_1\alpha_2\alpha_3\;,\; 1-\alpha_1\alpha_3\}\;.$$

Proof. a) By the hypotheses, it follows that for each $V_r \in \mathcal{V}$, with $r > 5$, there exists $h \in \{1, \ldots, 5\}$ such that $V_r \leq V_h$, therefore g_r is not relevant.
b) Let us assume $\alpha_1 + \alpha_2 \leq 1$. Then, the segment $(\alpha_1, \alpha_2, z), 0 \leq z \leq 1$, intersects the plane containing the triangle $V_2V_4V_5$ in the point $V^* = (\alpha_1, \alpha_2, z^*)$, with $z^* = \frac{1-\alpha_1-\alpha_2+\alpha_2\alpha_3}{1-\alpha_1} \leq \alpha_3$. Then $V^* \leq \mathcal{A}$. Moreover, for a suitable non negative vector $(\lambda_2, \lambda_4, \lambda_5)$, with $\lambda_2 + \lambda_4 + \lambda_5 = 1$, it is

$$V^* = \lambda_2 V_2 + \lambda_4 V_4 + \lambda_5 V_5 \ .$$

Then, $\mathcal{T} = \{2, 4, 5\}$ is a basic set. Moreover, the vector $(\lambda_2, \lambda_4, \lambda_5)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence $\mathcal{A}$ is g-coherent.
We observe that by the same reasoning it could be shown that $\mathcal{T} = \{3, 4, 5\}$ is a basic set too.
c) Let us assume that $\alpha_1 + \alpha_2 > 1$. Then, concerning the point $(\alpha_1, \alpha_2, z^*)$ belonging to the triangle $V_1V_4V_5$, one has

$$z^* = \alpha_3(2 - \alpha_1 - \alpha_2) < \alpha_3 \ ,$$

so that we don't need to consider such triangle. Then, it is enough to consider the equations of the planes containing the triangles

$$V_1V_2V_3 \ , \ \ V_1V_3V_4 \ , \ \ V_1V_2V_5 \ ,$$

which are respectively

$$\alpha_2 x + \alpha_1 y + (\alpha_1 + \alpha_2 - \alpha_1\alpha_2) z = \alpha_1 + \alpha_2 \ ,$$

$$(1 - \alpha_3 + \alpha_2\alpha_3) x + \alpha_3 y + z = 1 + \alpha_2\alpha_3 \ ,$$

$$\alpha_3(1 - \alpha_2) x + (1 - \alpha_3) y + (1 - \alpha_2) z = 1 - \alpha_2\alpha_3 \ .$$

The intersection points of the segment (α_1, α_2, z), $0 \leq z \leq 1$, with the planes above, are respectively

$$V_1^* = (\alpha_1, \alpha_2, z_1) \ , \ \ V_2^* = (\alpha_1, \alpha_2, z_2) \ , \ \ V_3^* = (\alpha_1, \alpha_2, z_3) \ ,$$

with

$$z_1 = \frac{\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1\alpha_2} \ , \ \ z_2 = 1 - \alpha_1 + \alpha_1\alpha_3 - \alpha_1\alpha_2\alpha_3 \ , \ \ z_3 = 1 - \alpha_1\alpha_3 \ .$$

Then, if the condition $\alpha_3 \leq Max\{z_1, z_2, z_3\}$ is satisfied, there exists a subscript $h \in \{1, 2, 3\}$ such that $V_h^* \geq \mathcal{A}$. Moreover, V_h^* belongs to one of the above triangles, say $V_{i_1}V_{i_2}V_{i_3}$. Then, for a suitable non negative vector $(\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_3})$, with $\lambda_{i_1} + \lambda_{i_2} + \lambda_{i_3} = 1$, it is

$$V_h^* = \lambda_{i_1} V_{i_1} + \lambda_{i_2} V_{i_2} + \lambda_{i_3} V_{i_3} \ .$$

Then, $\mathcal{T} = \{i_1, i_2, i_3\}$ is a basic set. Moreover, the vector $(\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_3})$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence $\mathcal{A}$ is g-coherent.
We observe that, if $\alpha_3 > Max\{z_1, z_2, z_3\}$, then the system $(\mathcal{S}_3^{\mathcal{T}})$ is not solvable and hence $\mathcal{A}$ is not g-coherent.
In conclusion, under the hypothesis $\alpha_1 + \alpha_2 > 1$, $\mathcal{A}$ is g-coherent iff $\alpha_3 \leq Max\{z_1, z_2, z_3\}$.

Remark 7 We observe that, under similar hypotheses, Theorem 16 can also be proved, obtaining similar results, in the following cases

$$V_1 = (1, 0, 1) \ , \quad \{V_2, V_3, V_4, V_5\} = \{(1, \alpha_2, 0), (\alpha_1, 1, 0), (0, 1, \alpha_3), (0, \alpha_2, 1)\} \ ;$$

$$V_1 = (0, 1, 1) \ , \quad \{V_2, V_3, V_4, V_5\} = \{(1, \alpha_2, 0), (1, 0, \alpha_3), (\alpha_1, 1, 0), (\alpha_1, 0, 1)\} \ .$$

Theorem 17 *If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = \emptyset$, $\mathcal{U}_2 = \{V_1, V_2, V_3, V_4, V_5, V_6\} = \{(1, \alpha_2, 0), (1, 0, \alpha_3), (\alpha_1, 1, 0), (0, 1, \alpha_3), (\alpha_1, 0, 1), (0, \alpha_2, 1))\}$, $\alpha_i < 1 \, \forall \, i$, then one has:*
a) for every $r > 6$, the gain g_r is not relevant;
b) if $\alpha_1 + \alpha_2 \leq 1$, or $\alpha_1 + \alpha_3 \leq 1$, or $\alpha_2 + \alpha_3 \leq 1$, then there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = 2$, and $\mathcal{A}$ is g-coherent.
c) if $\alpha_1 + \alpha_2 > 1$, $\alpha_1 + \alpha_3 > 1$, $\alpha_2 + \alpha_3 > 1$, then $\mathcal{A}$ is not g-coherent.

Proof. a) By the hypotheses, it follows that for each $V_r \in \mathcal{V}$, with $r > 6$, there exists $h \in \{1, \dots, 6\}$ such that $V_r \leq V_h$, therefore g_r is not relevant.
b.1) Let us assume $\alpha_1 + \alpha_2 \leq 1$. Then, defining $V^* = \alpha_1 V_2 + (1 - \alpha_1) V_4$, we have

$$V^* = (\alpha_1, 1 - \alpha_1, \alpha_3) \geq \mathcal{A} \, .$$

Then, $\mathcal{T} = \{2, 4\}$ is a basic set. Moreover, $(\alpha_1, 1 - \alpha_1)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence $\mathcal{A}$ is g-coherent.
b.2) Let us assume $\alpha_1 + \alpha_3 \leq 1$. Then, defining $V^* = \alpha_1 V_1 + (1 - \alpha_1) V_6$, we have

$$V^* = (\alpha_1, \alpha_2, 1 - \alpha_1) \geq \mathcal{A} \, .$$

Then, $\mathcal{T} = \{1, 6\}$ is a basic set. Moreover, $(\alpha_1, 1 - \alpha_1)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence $\mathcal{A}$ is g-coherent.

b.3) Let us assume $\alpha_2 + \alpha_3 \leq 1$. Then, defining $V^* = \alpha_2 V_3 + (1 - \alpha_2) V_5$, we have

$$V^* = (\alpha_1, \alpha_2, 1 - \alpha_2) \geq \mathcal{A} \, .$$

Then, $\mathcal{T} = \{3, 5\}$ is a basic set. Moreover, $(\alpha_2, 1 - \alpha_2)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence $\mathcal{A}$ is g-coherent.
c) Let us assume that $\alpha_1 + \alpha_2 > 1$, $\alpha_1 + \alpha_3 > 1$, $\alpha_2 + \alpha_3 > 1$. As an example, let us consider the intersection points, $(\alpha_1, \alpha_2, z^*)$ and $(\alpha_1, \alpha_2, z^{**})$, of the segment (α_1, α_2, z), $0 \leq z \leq 1$, with the triangles $V_1 V_4 V_5$ and $V_2 V_3 V_6$, respectively. It can be verified that

$$z^* = \frac{1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 + \alpha_2\alpha_3 - \alpha_1\alpha_2\alpha_3}{1 - \alpha_1 + \alpha_1\alpha_2} \, ,$$

$$z^{**} = \frac{1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 + \alpha_1\alpha_3 - \alpha_1\alpha_2\alpha_3}{1 - \alpha_2 + \alpha_1\alpha_2} \, .$$

Moreover, each one of the following conditions

$$z^* \geq \alpha_3 \, , \quad z^{**} \geq \alpha_3$$

is satisfied iff the following condition holds

$$\alpha_3 \leq \frac{1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2}{1 - \alpha_1 - \alpha_2 + 2\alpha_1\alpha_2} \, . \tag{11}$$

By hypothesis it is $\alpha_1 + \alpha_3 > 1$, $\alpha_2 + \alpha_3 > 1$. Therefore, if (11) were satisfied, it should be

$$\alpha_1 + \tfrac{1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2}{1 - \alpha_1 - \alpha_2 + 2\alpha_1\alpha_2} > 1 \, ,$$

$$\alpha_2 + \tfrac{1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2}{1 - \alpha_1 - \alpha_2 + 2\alpha_1\alpha_2} > 1 \, .$$

The inequalities above are satisfied iff $\alpha_2 < \frac{1}{2}$ and $\alpha_1 < \frac{1}{2}$, respectively. Then, it should be $\alpha_1 + \alpha_2 < 1$, which contradicts the hypothesis.
By a similar reasoning, it can be verified that for each $\{i, j, k\} \subseteq \{1, \dots, 6\}$, if the segment (α_1, α_2, z), $0 \leq z \leq 1$, intersects the triangle $V_i V_j V_k$ in a

point $V^* = (\alpha_1, \alpha_2, z^*)$, then one has $z^* < \alpha_3$.
Therefore, as $\alpha_1 + \alpha_2 > 1$, it follows

$$z^* < \alpha_3 \,, \quad z^{**} < \alpha_3 \,.$$

Then, the system $(\mathcal{S}_3^{\mathcal{T}})$ is not solvable and hence $\mathcal{A}$ is not g-coherent.

By a similar reasoning the following results can be proved.

Theorem 18 *If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = \emptyset$, $\mathcal{U}_2 = \{V_1, V_2, V_3, V_4\} = \{(1, \alpha_2, 0), (\alpha_1, 1, 0), (\alpha_1, 0, 1), (0, \alpha_2, 1))\}$, $\alpha_i < 1 \,\forall\, i$, then one has:*
a) for every $r > 4$, the gain g_r is not relevant;
b) if $\alpha_1 + \alpha_3 \leq 1$, or $\alpha_2 + \alpha_3 \leq 1$, then there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = 2$, and $\mathcal{A}$ is g-coherent.
c) if $\alpha_1 + \alpha_3 > 1$, $\alpha_2 + \alpha_3 > 1$, then $\mathcal{A}$ is not g-coherent.

Theorem 19 *If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = \emptyset$, $\mathcal{U}_2 = \{V_1, V_2, V_3, V_4\} = \{(1, 0, \alpha_3), (0, 1, \alpha_3), (\alpha_1, 1, 0), (\alpha_1, 0, 1))\}$, $\alpha_i < 1 \,\forall\, i$, then one has:*
a) for every $r > 4$, the gain g_r is not relevant;
b) if $\alpha_1 + \alpha_2 \leq 1$, or $\alpha_2 + \alpha_3 \leq 1$, then there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = 2$, and $\mathcal{A}$ is g-coherent.
c) if $\alpha_1 + \alpha_2 > 1$, $\alpha_2 + \alpha_3 > 1$, then $\mathcal{A}$ is not g-coherent.

Theorem 20 *If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = \emptyset$, $\mathcal{U}_2 = \{V_1, V_2, V_3, V_4\} = \{(1, 0, \alpha_3), (0, 1, \alpha_3), (1, \alpha_2, 0), (0, \alpha_2, 1))\}$, $\alpha_i < 1 \,\forall\, i$, then one has:*
a) for every $r > 4$, the gain g_r is not relevant;
b) if $\alpha_1 + \alpha_2 \leq 1$, or $\alpha_1 + \alpha_3 \leq 1$, then there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = 2$, and $\mathcal{A}$ is g-coherent.
c) if $\alpha_1 + \alpha_2 > 1$, $\alpha_1 + \alpha_3 > 1$, then $\mathcal{A}$ is not g-coherent.

Theorem 21 *If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = \mathcal{U}_2 = \emptyset$, $\mathcal{U}_3 = \{V_1, V_2, V_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1))\}$, $\alpha_i < 1 \,\forall\, i$, then one has:*
a) if $\alpha_1 + \alpha_2 \leq 1$, $\alpha_1 + \alpha_3 \leq 1$, $\alpha_2 + \alpha_3 \leq 1$, then $\mathcal{T} = \{1, 2, 3\}$ is a basic set;
b) $\mathcal{A}$ is g-coherent iff $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$.

Proof. a) We first show that the gains associated with the vectors $V_4 = (\alpha_1, \alpha_2, 0)$, $V_5 = (\alpha_1, 0, \alpha_3)$, $V_6 = (0, \alpha_2, \alpha_3)$ are not relevant. As $\alpha_1 + \alpha_2 \leq 1$, it follows that, for every pair (a, b), with

$$a > 0 \,,\; b > 0 \,, \quad a + b \leq 1 \,, \quad \frac{\alpha_2}{1 - \alpha_2} a \;\leq b \;\leq \frac{1 - \alpha_1}{\alpha_1} a \,,$$

it is $g_4 \leq a g_1 + b g_2$, therefore g_4 is not relevant.
As $\alpha_1 + \alpha_3 \leq 1$, it follows that, for every pair (a, b), with

$$a > 0 \,,\; b > 0 \,, \quad a + b \leq 1 \,, \quad \frac{\alpha_3}{1 - \alpha_3} a \;\leq b \;\leq \frac{1 - \alpha_1}{\alpha_1} a \,,$$

it is $g_5 \leq a g_1 + b g_3$, therefore g_5 is not relevant.
As $\alpha_2 + \alpha_3 \leq 1$, it follows that, for every pair (a, b), with

$$a > 0 \,,\; b > 0 \,, \quad a + b \leq 1 \,, \quad \frac{\alpha_3}{1 - \alpha_3} a \;\leq b \;\leq \frac{1 - \alpha_2}{\alpha_2} a \,,$$

it is $g_6 \leq a g_2 + b g_3$, therefore g_6 is not relevant.
For $r > 6$ there exists $h \in \{1, 2, 3\}$ such that $V_r \leq V_h$, so that g_r is not

relevant. Therefore $\mathcal{T} = \{1, 2, 3\}$ is a basic set.
b) If $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$, then the segment $(\alpha_1, \alpha_2, z), 0 \leq z \leq 1$, intersects the triangle $V_1V_2V_3$ in the point

$$V* = (\alpha_1, \alpha_2, z^*) = (\alpha_1, \alpha_2, 1-\alpha_1-\alpha_2) = \alpha_1 V_1 + \alpha_2 V_2 + (1-\alpha_1-\alpha_2) V_3 \geq \mathcal{A} .$$

Then, the vector $(\alpha_1, \alpha_2, 1 - \alpha_1 - \alpha_2)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence $\mathcal{A}$ is g-coherent.
If $\alpha_1 + \alpha_2 + \alpha_3 > 1$, then $\mathcal{A} > V^*$ and the system $(\mathcal{S}_3^{\mathcal{T}})$ is not solvable. Hence $\mathcal{A}$ is not g-coherent.

Remark 8 We observe that, given a vector of lower bounds $\mathcal{A} = (\alpha_1, \alpha_2)$ on the family $\mathcal{F} = \{E_1|H_1, E_2|H_2\}$, it is

$$\mathcal{V} \subseteq \{(1,1),\ (1,\alpha_2),\ (\alpha_1,1),\ (1,0),\ (0,1),\ (\alpha_1,0),\ (0,\alpha_2),\ (0,0)\} .$$

Then, assuming

$$\emptyset \neq E_iH_i \neq H_i\, ,\ \ 0 < \alpha_i < 1\, ,\ \ i = 1, 2\, ,$$

the following assertions can be easily proved:
(i) if $\{(1,1),\ (1,\alpha_2),\ (\alpha_1,1)\} \cap \mathcal{V} \neq \emptyset$, then $\mathcal{A}$ is g-coherent;
(ii) if $\{(1,1),\ (1,\alpha_2),\ (\alpha_1,1)\} \cap \mathcal{V} = \emptyset$, then $\{(1,0),\ (0,1)\} \subseteq \mathcal{V}$ and $\mathcal{A}$ is g-coherent iff $\alpha_1 + \alpha_2 \leq 1$.

4 Some general results

In this section we generalize some of the results given in the previous one.

Theorem 22 *If $\mathcal{V}_0 = \mathcal{V}_1 = \cdots = \mathcal{V}_{n-1} = \emptyset$ and $\alpha_1 + \cdots + \alpha_n > n - 1$, then $\mathcal{A}_n$ is not g-coherent.*

Proof. We consider the sets $N_r, r = 0, 1, \ldots, n-1$, defined in (8). Moreover, with each $V_r \in \mathcal{V}$ we associate the set

$$M_r = \{i \in J_n \,:\, v_{ri} = 0\} . \tag{12}$$

Then, we define the sets

$$\mathcal{U}_{h,k} = \{V_r \in \mathcal{V} \,:\, |N_r| = h\, ,\ |M_r| = k\}, \quad \begin{array}{l} h = 0, \ldots, n-1 \\ k = 1, \ldots, n\, . \end{array} \tag{13}$$

We observe that, if the sets $\mathcal{U}_{h,0}$ were defined, then it would be $\mathcal{V}_h = \mathcal{U}_{h,0}$. By the hypotheses, we have

$$\mathcal{V} = \bigcup_{h,k} \mathcal{U}_{h,k}$$

We recall that, the system $(\mathcal{S}_n)$ is solvable iff there exists a vector $\Lambda = (\lambda_1, \ldots, \lambda_m)$ such that

$$\sum_{h=1}^{m} \lambda_h V_h \geq \mathcal{A}_n\ ,\ \ \sum_{h=1}^{m} \lambda_h = 1\ ,\ \ \lambda_h \geq 0\ ,\ \forall\, h\, .$$

Let us define

$$V_1^* = (0, 1, \ldots, 1)\ ,\quad V_2^* = (1, 0, 1, \ldots, 1)\ , \ldots ,\quad V_n^* = (1, \ldots, 1, 0)\ .$$

We observe that for every non negative vector $(\lambda_1^*, \ldots, \lambda_n^*)$, with $\sum_i \lambda_i^* = 1$, for the vector

$$V^* = (v_1^*, \ldots, v_n^*) = \lambda_1^* V_1^* + \cdots + \lambda_n^* V_n^*$$

it is $\sum_i v_i^* = n-1$. Therefore $\mathcal{V}^* \geq \mathcal{A}_n$ implies $\sum_i \alpha_i \leq n-1$. Then, as for each $\mathcal{V}_r \in \mathcal{V}$ there exists $h \in J_n$ such that $V_r \leq V_h^*$, based on any solution $(\lambda_1, \ldots, \lambda_m)$ of $(\mathcal{S}_n)$, we would obtain

$$\lambda_1^* V_1^* + \cdots + \lambda_n^* V_n^* \geq \mathcal{A}_n \ , \tag{14}$$

with

$$\lambda_1^* \ = \ \sum_{r \in J_1^*} \lambda_r \ , \quad \cdots \ , \quad \lambda_n^* \ = \ \sum_{r \in J_n^*} \lambda_r \ ,$$

where $J_1^* = \{r \,:\, V_r \leq V_1^*\}$ and, for each $t = 2, \ldots, n$,

$$J_t^* = \{r \,:\, V_r \not\leq V_1^*, \ldots, V_r \not\leq V_{t-1}^*, V_r \leq V_t^*\} \ .$$

The proof follows by observing that, if the condition (14) were satisfied, then it would be $\alpha_1 + \cdots + \alpha_n \leq n-1$, which is a contradiction.

Theorem 23 *If* $\mathcal{V}_0 = \cdots = \mathcal{V}_{n-1} = \emptyset$, $|\mathcal{U}_{0,1}| = n$, $\alpha_i < 1 \, \forall \, i$, *then one has:*
a) there exists a basic set $\mathcal{T}$, *with* $|\mathcal{T}| = n$;
b) $\mathcal{A}_n$ *is g-coherent iff* $\alpha_1 + \cdots + \alpha_n \leq n-1$.

Proof. a) We represent $\mathcal{U}_{0,1}$ as the set $\{V_1, \ldots, V_n\}$, with

$$V_1 = (0, 1, \ldots, 1) \ , \quad V_2 = (1, 0, 1, \ldots, 1) \ , \ldots , \quad V_n = (1, \ldots, 1, 0) \ .$$

As $\mathcal{V}_0 = \cdots = \mathcal{V}_{n-1} = \emptyset$, for each $V_r \in \mathcal{V}$, with $r > n$, there exists a subscript i such that $v_{ri} = 0$. Then, there exists $h \in J_n$ such that $V_r \leq V_h$ and hence $g_r \leq g_h$, so that g_r is not relevant. Then, $\mathcal{T} = J_n = \{1, 2, \ldots, n\}$ is a basic set.
b.1) If $\alpha_1 + \cdots + \alpha_n > n-1$, then by Theorem 22 it follows that $\mathcal{A}$ is not g-coherent.
b.2) Let us assume that $\alpha_1 + \cdots + \alpha_{n-1} > n-1$. Then, it is

$$\alpha_n \leq n - 1 - \sum_{i=1}^{n-1} \alpha_i \ .$$

Moreover, assuming that $\alpha_1 + \cdots + \alpha_{n-1} > n-2$, let us consider the (positive) quantities

$$\lambda_i = 1 - \alpha_i \ , \ \ i \in J_{n-1} \ ; \ \ \lambda_n = 1 - \sum_{i \in J_{n-1}} \lambda_i = \sum_{i \in J_{n-1}} \alpha_i - (n-2) \ .$$

Given the vector $V^* = (v_1^*, \ldots, v_n^*)$ defined as

$$V^* = \lambda_1 V_1 + \cdots + \lambda_n V_n \ ,$$

we have

$$v_i^* = \textstyle\sum_{j \in J_n \setminus \{i\}} \lambda_j = 1 - \lambda_i = \alpha_i \ , \ \ i \in J_{n-1} \, ,$$

$$v_n^* = \textstyle\sum_{j \in J_{n-1}} \lambda_j = 1 - \lambda_n = n - 1 - \sum_{\in J_{n-1}} \alpha_i \geq \alpha_n.$$

Therefore $V^* \geq \mathcal{A}_n$. Then, the vector $(\lambda_1, \ldots, \lambda_n)$ is a solution of the system $(\mathcal{S}_n^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence $\mathcal{A}_n$ is g-coherent.
b.2) Let us assume that $\alpha_1 + \cdots + \alpha_{n-1} \leq n-2$. Then, denote by $k \in J_{n-2}$ the minimum integer such that the following conditions (i) and (ii) hold:
(i) $\alpha_1 + \cdots + \alpha_k > k-1$;
(ii) $\alpha_1 + \cdots + \alpha_{k+1} \leq k$.
Moreover, consider the (positive) quantities

$$\lambda_i = 1 - \alpha_i \ , \ \ i \in J_k \ , \ \ \lambda_{k+1} = \sum_{i \in J_k} \alpha_i - (k-1) \, .$$

We observe that $\lambda_1 + \cdots + \lambda_{k+1} = 1$. Moreover, considering the vector

$$V^* = (v_1^*, \ldots, v_n^*) = \lambda_1 V_1 + \cdots + \lambda_{k+1} V_{k+1} \ ,$$

it can be verified that

$$v_i^* = \textstyle\sum_{j \in J_{k+1} \setminus \{i\}} \lambda_j = 1 - \lambda_i = \alpha_i \ , \ \ i \in J_{k+1} \ ,$$

$$v_i^* = 1 \ , \ \ k+1 < i \leq n \, .$$

Then, it is $V^* \geq \mathcal{A}_n$ and the vector $(\lambda_1, \ldots, \lambda_{k+1}, 0, \ldots, 0)$ is a solution of the system $(\mathcal{S}_n^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$. Therefore, $\mathcal{A}_n$ is g-coherent.
We denote by $\mathcal{Z}$ the set defined as

$$\mathcal{Z} = \{(h,k) \, : \, h+k = n-1 \ , \ h > 0\} \cup \{(h,k) \, : \, h+k < n-1\}$$

Theorem 24 *If $\mathcal{V}_0 = \cdots = \mathcal{V}_{n-1} = \emptyset$, $\mathcal{U}_{h,k} = \emptyset$ for each $(h,k) \in \mathcal{Z}$, and $\alpha_1 + \cdots + \alpha_n > 1$, then $\mathcal{A}_n$ is not g-coherent.*

Proof. By the hypotheses, one has

$$\mathcal{V} = \mathcal{U}_{0,n-1} \cup (\bigcup_{h=0}^{n-1} \mathcal{U}_{h,n-h}) \, .$$

Given a nonnegative vector $\Lambda = (\lambda_1, \ldots, \lambda_m)$, with $\sum_{r \in J_m} \lambda_r = 1$, let us consider the vector

$$V^* = (v_1^*, \ldots, v_n^*) = \sum_{r \in J_m} \lambda_r V_r \, .$$

One has

$$V^* = \sum_{r : V_r \in \mathcal{U}_{0,n-1}} \lambda_r V_r + \sum_{r : V_r \notin \mathcal{U}_{0,n-1}} \lambda_r V_r = V_1^* + V_2^* \, ,$$

where

$$V_1^* = (v_{11}^*, \ldots, v_{1n}^*) = \textstyle\sum_{r : V_r \in \mathcal{U}_{0,n-1}} \lambda_r V_r \ ,$$

$$V_2^* = (v_{21}^*, \ldots, v_{2n}^*) = \textstyle\sum_{r : V_r \notin \mathcal{U}_{0,n-1}} \lambda_r V_r \, .$$

Let us consider the quantity $\delta = \sum_{r : V_r \in \mathcal{U}_{0,n-1}} \lambda_r$. We observe that for each $V_r \in \mathcal{U}_{h,n-h}$, $0 \leq h \leq n-1$, it is $V_r \leq \mathcal{A}$ and hence $V_2^* \leq \mathcal{A}$. Then, if $\delta = 0$, one has

$$V^* = V_2^* \leq \mathcal{A} \, .$$

Therefore, doesn't exist a nonnegative vector Λ, with $\sum_{r \in J_m} \lambda_r = 1$ and $\delta = 0$, such that

$$\lambda_1 V_1 + \cdots + \lambda_m V_m \geq \mathcal{A} \, .$$

Then, under the condition $\delta = 0$, the system $(\mathcal{S}_n^{\mathcal{T}})$ is not solvable and hence $\mathcal{A}_n$ is not g-coherent.
If $\delta > 0$, then

$$V^* = V_1^* + V_2^* = \delta \overline{V}_1 + V_2^* \, ,$$

where

$$\overline{V}_1 = (\overline{v}_{11}, \dots, \overline{v}_{1n}) = \sum_{r: V_r \in \mathcal{U}_{0,n-1}} \overline{\lambda}_r V_r \, ,$$

$$\overline{\lambda}_r = \frac{\lambda_r}{\delta} \, , \;\; \forall \, r \, : \, V_r \in \mathcal{U}_{0,n-1} \, .$$

Notice that

$$\overline{v}_{2i} \leq (1-\delta)\alpha_i \, , \;\; \forall \, i \in J_n \, .$$

Moreover,

$$\sum_{i \in J_n} \overline{v}_{1i} = 1 < \sum_{i \in J_n} \alpha_i \, ,$$

and hence there exists a subscript h such that $\overline{v}_{1h} < \alpha_h$. Then,

$$v_h^* = \delta \overline{v}_{1h} + \overline{v}_{2h} < \delta \alpha_h + (1-\delta)\alpha_h = \alpha_h \, ,$$

therefore $V^* = \sum_{r \in J_m} \lambda_r V_r \not\geq \mathcal{A}$. Then, under the condition $\delta > 0$, the system $(\mathcal{S}_n^{\mathcal{T}})$ is still not solvable. In conclusion, under the hypothesis $\alpha_1 + \cdots + \alpha_n > 1$, the system $(\mathcal{S}_n^{\mathcal{T}})$ is not solvable and hence $\mathcal{A}_n$ is not g-coherent.

Theorem 25 *If $\mathcal{V}_0 = \cdots = \mathcal{V}_{n-1} = \emptyset$, $\mathcal{U}_{h,k} = \emptyset$, for each pair $(h,k) \in \mathcal{Z}$, $|\mathcal{U}_{0,n-1}| = n$, $\alpha_i < 1 \, \forall \, i$, then one has:*
a) if, for every $j \in J_n$, it is $\sum_{i \in J_n \setminus \{j\}} \alpha_i \leq 1$, then $\mathcal{T} = J_n$ is a basic set;
b) $\mathcal{A}_n$ is g-coherent iff $\alpha_1 + \cdots + \alpha_n \leq 1$.

Proof. a) Let us assume that, for every $j \in J_n$, it is $\sum_{i \in J_n \setminus \{j\}} \alpha_i \leq 1$. Moreover, define

$$\mathcal{U}_{0,n-1} = \{V_1, \dots, V_n\} \, ,$$

where

$$V_1 = (1, 0, \dots, 0) \, , \; \dots, V_n = (0, \dots, 0, 1) \, .$$

We observe that for each $V_r \in \mathcal{U}_{h,n-h}$, with $0 \leq h \leq n-1$, it is

$$v_{ri} = \alpha_i \, , \; i \in N_r \, , \;\; v_{ri} = 0 \, , \; i \in M_r = J_n \setminus N_r \, , \; N_r \subset J_n \, , \; M_r \neq \emptyset \, ,$$

and hence

$$V_r = \sum_{i \in N_r} \alpha_i V_i \, .$$

Then,

$$V_r - \mathcal{A} = \sum_{i \in N_r} \alpha_i (V_i - \mathcal{A}) + (\sum_{i \in N_r} \alpha_i - 1)\mathcal{A} \, .$$

Recalling (1) and defining the linear function

$$f(z_1, \dots, z_n) = \sum_{i=1}^n s_i z_i \, , \;\; s_i \geq 0 \, , \; \forall \, i \in J_n \, ,$$

for every $r \in J_m$ we have

$$g_r = f(V_r - \mathcal{A}_n) \, .$$

Then, observing that

$$\sum_{i\in N_r}\alpha_i - 1 \le 0\ ,\quad f(\mathcal{A}) = \sum_{i=1}^{n} s_i\alpha_i \ge 0\ ,$$

one has

$$g_r = f(V_r - \mathcal{A}) = \sum_{i\in N_r}\alpha_i f(V_i - \mathcal{A}) + (\sum_{i\in N_r}\alpha_i - 1) f(\mathcal{A}) \le$$
$$\le \sum_{i\in N_r}\alpha_i f(V_i - \mathcal{A}) = \sum_{i\in N_r}\alpha_i g_i\ .$$

Then, g_r is not relevant and hence $\mathcal{T} = J_n$ is a basic set.
b.1) If $\alpha_1 + \cdots + \alpha_n > 1$, then by Theorem 24 it follows that $\mathcal{A}$ is not g-coherent.
b.2) Let us assume $\alpha_1 + \cdots + \alpha_n \le 1$. Then, $\alpha_n \le 1 - \sum_{i=1}^{n-1}\alpha_i$. The segment

$$(\alpha_1, \ldots, \alpha_{n-1}, z)\ ,\quad 0 \le z \le 1\ ,$$

intersects the hyperplane $\alpha_1 + \cdots + \alpha_n \le 1$ in the point $V^* = (\alpha_1, \ldots, \alpha_{n-1}, z^*)$, with $z^* = 1 - \sum_{i=1}^{n-1}\alpha_i \ge \alpha_n$ and hence $V^* \ge \mathcal{A}$. Moreover,

$$V^* = \alpha_1 V_1 + \cdots + \alpha_{n-1}V_{n-1} + (1 - \sum_{i=1}^{n-1}\alpha_i)V_n\ .$$

Then, the vector $(\alpha_1, \ldots, \alpha_{n-1}, 1 - \sum_{i=1}^{n-1}\alpha_i)$ is a solution of the system $(\mathcal{S}_n^{\mathcal{T}})$, with $|I_0^{\mathcal{T}}| \le 1$ and hence $\mathcal{A}_n$ is g-coherent.

5 Deciding g-coherence

We denote respectively by $\Lambda = (\lambda_1, \ldots, \lambda_m)$ and S the vector of unknowns and the set of solutions of the system $(\mathcal{S}_n)$. Moreover, given a subset $S' \subseteq S$, for each $j \in J_n$ we consider the linear function

$$\Phi_j(\Lambda) = \Phi_j(\lambda_1, \ldots, \lambda_m) = \sum_{r:C_r\subseteq H_j}\lambda_r\ .$$

Then, we denote by I_0' be the set defined as

$$I_0' = \{j \in J_n\ :\ Max_{\Lambda\in S'}\Phi_j(\Lambda) = 0\} \tag{15}$$

and by $(\mathcal{F}_0', \mathcal{A}_0')$ the pair associated with I_0'. Then, the following result has been proved in [2].

Theorem 26 *Given $S' \subseteq S$, the imprecise assessment $\mathcal{A}_n$ on $\mathcal{F}_n$ is g-coherent if and only if the following conditions are satisfied:*

1. *the system $(\mathcal{S}_n)$ is solvable ;*
2. *if $I_0' \neq \emptyset$, then $\mathcal{A}_0'$ is g-coherent.*

Given the set of constituents $\mathcal{C} = \{C_h\ ,\ h \in J_m\}$ and a subset $\Gamma \subset J_n$, let us consider the subset $J_\Gamma \subseteq J_m$ defined as

$$J_\Gamma = \{r \in J_m\ :\ C_r \subseteq H_j^c, \forall\, j \notin \Gamma\}\ . \tag{16}$$

We observe that $|J_\Gamma| \le 3^{|\Gamma|} - 1$. We denote by $\mathcal{V}_\Gamma \subseteq \mathcal{V}$ the subset of vectors associated with J_Γ and by I_0^Γ the set obtained by (15), with $S' = S_\Gamma$. Notice that $J_n \setminus \Gamma \subseteq I_0^\Gamma$. Moreover, we denote by $(\mathcal{F}_0^\Gamma, \mathcal{A}_0^\Gamma)$ the pair associated with Γ. Then, consider the subset $S_\Gamma \subseteq S$ defined as

$$S_\Gamma = \{\Lambda \in S\ :\ \lambda_r = 0, \forall\, r \notin J_\Gamma\}\ .$$

Then, we have

Theorem 27 *Let us assume that $J_\Gamma \neq \emptyset$, $S_\Gamma \neq \emptyset$. Then, we have:*
(i)if $I_0^\Gamma = \emptyset$, then $\mathcal{A}_n$ is g-coherent;
(ii) if $I_0^\Gamma \neq \emptyset$, then $\mathcal{A}_n$ is g-coherent iff $\mathcal{A}_0^\Gamma$ is g-coherent.

Proof. We observe that, if $S_\Gamma \neq \emptyset$, then the system $(\mathcal{S}_n)$ is solvable.
(i) if $I_0^\Gamma = \emptyset$, then the g-coherence of $\mathcal{A}_n$ amounts to solvability of $(\mathcal{S}_n)$.
(ii) if $I_0^\Gamma \neq \emptyset$ and $\mathcal{A}_0^\Gamma$ is g-coherent, then applying Theorem 26, with $I_0' = I_0^\Gamma$, one has that $\mathcal{A}_n$ is g-coherent. Of course, if $\mathcal{A}_n$ is g-coherent, then $\mathcal{A}_0^\Gamma$ is g-coherent.

Based on Theorem 27 and on the results of the previous section, we could search for a subset $J_\Gamma \neq \emptyset$, with $|\Gamma| \leq 3$. We denote by $(\mathcal{S})$ the starting system associated with a pair $(\mathcal{F}, \mathcal{A})$, and by $(\mathcal{S}^\Gamma)$ the system obtained by $(\mathcal{S})$, with the added constraints

$$\lambda_r = 0 \; , \;\; \forall\, r \notin J_\Gamma \; .$$

Of course, the set of solutions of $(\mathcal{S}^\Gamma)$ is S_Γ. We denote by $\mathcal{K} = \{\Gamma_h \, , \; h = 1, \ldots, M\}$, where M is a suitable integer, the class of subsets Γ, with $|\Gamma| \leq 3$, such that $J_\Gamma \neq \emptyset$. Given as input the pair $(\mathcal{F}, \mathcal{A}) = (\mathcal{F}_n, \mathcal{A}_n)$ the following procedure, called $SubF\mathcal{K}()$, improves the procedure $SubF\mathcal{V}()$ by using the class $\mathcal{K}$ and returns a new pair $(\mathcal{F}, \mathcal{A}) = SubF\mathcal{K}(\mathcal{F}_n, \mathcal{A}_n)$.

Algorithm 3 $SubF\mathcal{K}(\mathcal{F}, \mathcal{A})$

1. Set $(\mathcal{F}, \mathcal{A})$= $SubF\mathcal{V}(\mathcal{F}, \mathcal{A})$.
2. If $(\mathcal{F}, \mathcal{A})$=$(\emptyset, \emptyset)$, then go to step 6
   else determine the class $\mathcal{K}$.
3. If $\mathcal{K} = \emptyset$, then go to step 6
   else set $h = 1$.
4. Construct the system $(\mathcal{S}^{\Gamma_h})$.
5. If $(\mathcal{S}^{\Gamma_h})$ is solvable, then compute the set $(I_0^{\Gamma_h})$. Set $(\mathcal{F}, \mathcal{A}) = (\mathcal{F}_0^{\Gamma_h}, \mathcal{A}_0^{\Gamma_h})$ and go to Step 1.
   If $(\mathcal{S}^{\Gamma_h})$ is not solvable and $h < M$, then set $h = h+1$ and go to Step 4.
   If $(\mathcal{S}^{\Gamma_h})$ is not solvable and $h = M$, then go to step 6.
6. return $(\mathcal{F}, \mathcal{A})$.

Based on Theorem 2 and on the previous results, the checking of g-coherence can be made by the algorithm below.

Algorithm 4 *CheckgC*$(\mathcal{F}, \mathcal{A})$.

1. Set $(\mathcal{F}, \mathcal{A}) = SubF\mathcal{K}(\mathcal{F}, \mathcal{A})$.
2. If $(\mathcal{F}, \mathcal{A}) = (\emptyset, \emptyset)$ return (*true*).
3. Determine a subset $\mathcal{T}$ satisfying, for all $r \notin \mathcal{T}$, the condition (5), with $\mathcal{T}_r \subseteq \mathcal{T}$.
4. Construct the system $(\mathcal{S}^\mathcal{T})$.
   If the system $(\mathcal{S}^\mathcal{T})$ is not solvable then return (*false*).
5. Compute the set $I_0^\mathcal{T}$.
   If $I_0^\mathcal{T} \neq \emptyset$ then set $(\mathcal{F}, \mathcal{A}) = (\mathcal{F}_0^\mathcal{T}, \mathcal{A}_0^\mathcal{T})$ and go to step 1,
   else return (*true*).

6 Some examples

We illustrate the results of previous sections by examining some examples.

Example 1 Given the family

$$\mathcal{K}_3 = \{B|A,\ C|AB,\ D|C\},$$

let us consider the vector of upper bounds

$$\mathcal{B}_3 = (0.2,\ 0.2,\ 0.2)$$

on $\mathcal{K}_3$. We observe that it is equivalent to consider the vector of lower bounds

$$\mathcal{A}_3 = (0.8,\ 0.8,\ 0.8)$$

on

$$\mathcal{F}_3 = \{B^c|A,\ C^c|AB,\ D^c|C\}\,.$$

The constituents contained in $\mathcal{H}_3 = A \vee C$ are respectively

$$C_1 = AB^cCD^c\ ,\quad C_2 = AB^cC^c\ ,\quad C_3 = A^cCD^c\ ,\quad C_4 = AB^cCD\ ,$$

$$C_5 = ABC^c\ ,\quad C_6 = ABCD^c\ ,\quad C_7 = A^cCD\ ,\quad C_8 = ABCD\ .$$

The associated vectors are:

$$V_1 = (1, 0.8, 1)\ ,\quad V_2 = (1, 0.8, 0.8)\ ,\quad V_3 = (0.8, 0.8, 1)\ ,\quad V_4 = (1, 0.8, 0)\ ,$$

$$V_5 = (0, 1, 0.8)\ ,\quad V_6 = (0, 0, 1)\ ,\quad V_7 = (0.8, 0.8, 0)\ ,\quad V_8 = (0, 0, 0)\ .$$

In our case, it is

$$\mathcal{V} = \mathcal{V}_0\ \cup\ \mathcal{V}_1\ \cup\ \mathcal{V}_2\ \cup\ \mathcal{U}_2\ \cup\ \mathcal{U}_3\ \cup\ \mathcal{U}_4\ \cup\ \mathcal{U}_6\,,$$

with

$$\mathcal{V}_0 = \emptyset\,,\ \ \mathcal{V}_1 = \{V_1\}\,,\ \ \mathcal{V}_2 = \{V_2, V_3\}\,,$$

$$\mathcal{U}_2 = \{V_4, V_5\}\,,\ \ \mathcal{U}_3 = \{V_6\}\,,\ \ \mathcal{U}_4 = \{V_7\}\,,\ \ \mathcal{U}_6 = \{V_8\}\,.$$

Then, by Theorem 8, $\mathcal{A}_3$ is g-coherent.We observe that, as $|\mathcal{V}_2| = 2$, the same conclusion would also follow by Theorem 6.

Example 2 Let be given the interval-valued assessment

$$([\frac{1}{5}, \frac{1}{4}],\ [\frac{1}{10}, \frac{1}{5}],\ [\frac{1}{10}, \frac{1}{4}])$$

on the family

$$\{B|AC,\ C|(A \vee B),\ D|(B \vee C)\}\,.$$

It is equivalent to consider the vector of lower bounds

$$\mathcal{A}_6 = (\frac{1}{5}, \frac{1}{10}, \frac{1}{10}, \frac{3}{4}, \frac{4}{5}, \frac{3}{4})$$

on

$$\mathcal{F}_6 = \{B|AC,\, C|(A \vee B),\, D|(B \vee C),\, B^c|AC,\, C^c|(A \vee B),\, D^c|(B \vee C)\}\,.$$

The constituents contained in $\mathcal{H}_6 = A \vee B \vee C$ are

$$C_1 = ABCD\,, \quad C_2 = ABCD^c\,, \quad C_3 = BC^cD\,, \quad C_4 = BC^cD^c\,,$$
$$C_5 = AB^cCD\,, \quad C_6 = AB^cCD^c\,, \quad C_7 = AB^cC^c\,, \quad C_8 = A^cBCD\,,$$
$$C_9 = A^cBCD^c\,, \quad C_{10} = A^cB^cCD\,, \quad C_{11} = A^cB^cCD^c\,.$$

The associated vectors are:

$$V_1 = (1,1,1,0,0,0)\,, \quad V_2 = (1,1,0,0,0,1)\,, \quad V_3 = (\tfrac{1}{5},0,1,\tfrac{3}{4},1,0)\,,$$
$$V_4 = (\tfrac{1}{5},0,0,\tfrac{3}{4},1,1)\,, \quad V_5 = (0,1,1,1,0,0)\,, \quad V_6 = (0,1,0,1,0,1)\,,$$
$$V_7 = (\tfrac{1}{5},0,\tfrac{1}{10},\tfrac{3}{4},1,\tfrac{3}{4})\,, \quad V_8 = (\tfrac{1}{5},1,1,\tfrac{3}{4},0,0)\,, \quad V_9 = (\tfrac{1}{5},1,0,\tfrac{3}{4},0,1)\,,$$
$$V_{10} = (\tfrac{1}{5},\tfrac{1}{10},1,\tfrac{3}{4},\tfrac{4}{5},0)\,, \quad V_{11} = (\tfrac{1}{5},\tfrac{1}{10},0,\tfrac{3}{4},\tfrac{4}{5},1)\,.$$

Notice that, for every $h = 0, 1, \ldots, 5$, it is $\mathcal{V}_h = \emptyset$. Moreover,

$$\mathcal{V} = \mathcal{U}_{0,3} \cup \mathcal{U}_{2,2} \cup \mathcal{U}_{4,1}\,,$$

with

$$\mathcal{U}_{0,3} = \{V_1, V_2, V_5, V_6\}\,, \;\; \mathcal{U}_{2,2} = \{V_3, V_4, V_8, V_9\}\,, \;\; \mathcal{U}_{4,1} = \{V_7, V_{10}, V_{11}\}\,.$$

Let us consider the subset (of J_6) $\Gamma = \{3, 6\}$ and the associated subset (of J_{11}) $J_\Gamma = \{10, 11\}$. We observe that, in our case, the subset S_Γ associated with J_Γ is

$$S_\Gamma = \{\Lambda \in S \,:\, \lambda_r = 0, \,\forall\, r < 10\}\,,$$

which is not empty iff $\alpha_3 + \alpha_6 \leq 1$. As $\alpha_3 + \alpha_6 = \frac{1}{10} + \frac{3}{4} = \frac{17}{20}$, one has $S_\Gamma \neq \emptyset$. Moreover, $I_0^\Gamma = I_6 \setminus \Gamma = \{1, 2, 4, 5\}$. Then, based on Theorem 27, we consider the pair $(\mathcal{F}_0^\Gamma, \mathcal{A}_0^\Gamma)$, with

$$\mathcal{F}_0^\Gamma = \mathcal{F}_4 = \{B|AC,\; C|(A \vee B),\; B^c|AC,\; C^c|(A \vee B)\}\,,$$
$$\mathcal{A}_0^\Gamma = \mathcal{A}_4 = (\tfrac{1}{5}, \tfrac{1}{10}, \tfrac{3}{4}, \tfrac{4}{5})\,.$$

Now, it is $\mathcal{H}_4 = A \vee B$ and the constituents contained in it are

$$C_1 = ABC\,,\; C_2 = BC^c\,,\; C_3 = AB^cC\,,\; C_4 = A^cBC\,,\; C_5 = AB^cC^c\,.$$

The associated vectors are

$$V_1 = (1,1,0,0),\; V_2 = V_5 = (\alpha_1, 0, \alpha_4, 1),\; V_3 = (0,1,1,0),\; V_4 = (\alpha_1, 1, \alpha_4, 0).$$

We have $\Gamma = \{2, 4\}$ and $J_\Gamma = \{2, 4, 5\}$. We observe that $S_\Gamma \neq \emptyset$ iff $\alpha_2 + \alpha_4 \leq 1$. As $\alpha_2 + \alpha_4 = \frac{1}{10} + \frac{4}{5} = \frac{9}{10}$, then $S_\Gamma \neq \emptyset$. Moreover, $I_0^\Gamma = J_4 \setminus \Gamma = \{1, 3\}$. Finally, as $\alpha_1 + \alpha_3 = \frac{1}{5} + \frac{3}{4} = \frac{19}{20}$, the condition $\alpha_1 + \alpha_3 \leq 1$ is satisfied and hence the assessment $\mathcal{A}_0^\Gamma = (\frac{1}{5}, \frac{3}{4})$ on $\mathcal{F}_0^\Gamma = \{B|AC,\; B^c|AC\}$ is g-coherent. Then, the initial assessment $\mathcal{A}_6$ on $\mathcal{F}_6$ is g-coherent too.

Example 3 This example is inspired by another one, examined under a different perspective in [13], which is a modified version of an example given in [19]. We consider a probabilistic knowledge base consisting of some conditional assertions, which concern a given party having various attributes (the party is *great*; the party is *noisy*; *Linda* and *Steve* are *present*; and so

on). By the symbol $A \mid\!\sim_{\varepsilon} B$ we denote the assessment $P(B|A) \geq \alpha$, where $\alpha = 1 - \varepsilon$.
We start with the set $\{G, L, N, S\}$, which is a shorthand notation for the set of (logically independent) events defined as:

(i) $G =$ "The party will be great",

(ii) $L =$ "Linda goes to the party",

(iii) $S =$ "Steve goes to the party",

(iv) $N =$ "The party will be noisy",

and with a probabilistic knowledge base which has the following rules and $\varepsilon-$values:

$$L \mid\!\sim_{0.05} G\,, \qquad L \mid\!\sim_{0.6} \neg S\,, \quad L \wedge S \mid\!\sim_{0.1} \neg N\,,$$

$$N \mid\!\sim_{0.4} (L \vee S)\,, \quad N \mid\!\sim_{0.3} G\,, \quad Linda \mid\!\sim_{0.4} \neg Noisy \wedge S\,.$$

With the above $\varepsilon-$values it is associated the vector of lower bounds

$$\mathcal{A}_6 = (\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4,\, \alpha_5,\, \alpha_6) = (0.95, 0.4, 0.9, 0.6, 0.7, 0.6)\ ,$$

defined on the family

$$\mathcal{F}_6 = \{G|L,\ S^c|L,\ N^c|LS,\ (L \vee S)|N,\ G|N,\ N^cS|L\}\ .$$

We want to study the g-coherence of $\mathcal{A}_5$.
It is $\mathcal{H}_5 = L \vee N$. The constituents contained in $\mathcal{H}_5$ are:

$$C_1 = LGSN, \quad C_2 = LGSN^c, \quad C_3 = LGS^cN, \quad C_4 = LGS^cN^c,$$

$$C_5 = LG^cSN, \quad C_6 = LG^cSN^c, \quad C_7 = LG^cS^cN, \quad C_8 = LG^cS^cN^c,$$

$$C_9 = L^cGSN, \quad C_{10} = L^cGS^cN, \quad C_{11} = L^cG^cSN, \quad C_{12} = L^cG^cS^cN.$$

The associated vectors are:

$$\begin{array}{ll}
V_1 = (1,0,0,1,1,0), & V_2 = (1,0,1,\alpha_4,\alpha_5,1), \\
V_3 = (1,1,\alpha_3,1,1,0), & V_4 = (1,1,\alpha_3,\alpha_4,\alpha_5,0), \\
V_5 = (0,0,0,1,0,0), & V_6 = (0,0,1,\alpha_4,\alpha_5,1), \\
V_7 = (0,1,\alpha_3,1,0,0), & V_8 = (0,1,\alpha_3,\alpha_4,\alpha_5,0), \\
V_9 = (\alpha_1,\alpha_2,\alpha_3,1,1,\alpha_6), & V_{10} = (\alpha_1,\alpha_2,\alpha_3,0,1,\alpha_6) \\
V_{11} = (\alpha_1,\alpha_2,\alpha_3,1,0,\alpha_6), & V_{12} = (\alpha_1,\alpha_2,\alpha_3,0,0,\alpha_6).
\end{array}$$

We first observe that $\mathcal{V}_4 = \{V_9\}$, with $N_9 = \{1, 2, 3, 6\}$. Then, $\mathcal{A}_6$ is g-coherent iff $\mathcal{A}_{N_9}$ is g-coherent. We have

$$\mathcal{F}_{N_9} = \mathcal{F}_4 = \{G|L,\ S^c|L,\ N^c|LS,\ N^cS|L\}\,,$$

$$\mathcal{A}_{N_9} = \mathcal{A}_4 = (0.95, 0.4, 0.9, 0.6)\,.$$

It is $\mathcal{H}_4 = L$ and the constituents contained in it are:

$$C_1 = LGSN, \quad C_2 = LGSN^c, \quad C_3 = LGS^c,$$

$$C_4 = LG^cSN, \quad C_5 = LG^cSN^c, \quad C_6 = LG^cS^c.$$

The associated vectors are:

$$V_1 = (1,0,0,0), \quad V_2 = (1,0,1,1), \quad V_3 = (1,1,\alpha_3,0),$$

$$V_4 = (0,0,0,0), \quad V_5 = (0,0,1,1), \quad V_6 = (0,1,\alpha_3,0),$$

We can only consider the subset (of J_4) $\Gamma = \{1,2,4\}$ with the associated subset (of J_6) $J_\Gamma = \{3,6\}$. The system $(\mathcal{S}^\Gamma)$, given below

$$\begin{cases} \lambda_3 \geq \alpha_1(\lambda_3 + \lambda_6)\,, \\ \lambda_3 + \lambda_6 \geq \alpha_2(\lambda_3 + \lambda_6)\,, \\ 0 \geq \alpha_4(\lambda_3 + \lambda_6)\,, \\ \lambda_3 + \lambda_6 = 1\,, \lambda_3 \geq 0\,,\ \lambda_6 \geq 0\,, \end{cases}$$

is not solvable. Then, to check the g-coherence of $\mathcal{A}_4$, we can determine a basic set $\mathcal{T}$ by studying the associated system $(\mathcal{S}_4^{\mathcal{T}})$. We observe that

$$V_2 > V_1 \geq V_4\,,\ \ V_2 \geq V_5\,,\ \ V_3 \geq V_6\,,$$

hence g_1, g_4, g_5, g_6 are not relevant and $\mathcal{T} = \{2,3\}$ is a basic set. We obtain the following system $(\mathcal{S}_4^{\mathcal{T}})$

$$\begin{cases} \lambda_2 + \lambda_3 \geq \alpha_1(\lambda_2 + \lambda_3)\,, \\ \lambda_3 \geq \alpha_2(\lambda_2 + \lambda_3)\,, \\ \lambda_2 \geq \alpha_3\lambda_2\,, \\ \lambda_2 \geq \alpha_4(\lambda_2 + \lambda_3)\,, \\ \lambda_2 + \lambda_3 = 1\,, \lambda_2 \geq 0\,, \lambda_3 \geq 0\,, \end{cases}$$

that is

$$1 \geq \alpha_1\,, \quad \lambda_3 \geq \alpha_2\,, \quad 1 \geq \alpha_3\,, \quad \lambda_2 \geq \alpha_4\,, \quad \lambda_2 + \lambda_3 = 1\,, \quad \lambda_2 \geq 0\,, \lambda_3 \geq 0\,.$$

The vector $\Lambda = (\lambda_2, \lambda_3) = (0.6, 0.4)$ is the unique solution of $(\mathcal{S}_4^{\mathcal{T}})$. Moreover, $I_0^{\mathcal{T}} = \emptyset$, therefore $\mathcal{A}_4$ is g-coherent and the initial assessment $\mathcal{A}_6$ is g-coherent too.

7 Conclusions

Exploiting some results and algorithms given in [1], [5], [6], we illustrated a probabilistic approach to uncertain reasoning, based on lower conditional probability bounds. Within our approach, the checking of g-coherence can be worked out with a reduced set of variables and a reduced set of linear constraints. To achieve such reduction, in each iteration our procedure determines a basic set of variables $\mathcal{T}$, by eliminating a subset of "not-relevant" gains. Moreover, using a suitable partition of the set of vectors $\{V_1, \ldots, V_m\}$ associated with the set of constituents $\{C_1, \ldots, C_m\}$, our algorithms can also reduce the number of linear constraints. In the paper, we have studied in detail imprecise assessments defined on families of three conditional events. We obtained some necessary and sufficient conditions of g-coherence and we also generalized some theoretical results. Exploiting such results, we proposed two algorithms which provide new strategies for reducing the number of constraints and for deciding g-coherence. We illustrated our procedures by examining some examples. Our approach could be combined with that ones given in [7], [8], where some logical conditions are studied with the aim of splitting the problem into suitable sub-problems, or [14], [15], where efficient techniques based on "column generation methods" are exploited, or [3], [4], where it is shown that probabilistic reasoning under coherence can be based on a combination of reasoning in probabilistic logic and default reasoning techniques. We notice that the computation in each iteration of the basic set $\mathcal{T}$ may be time consuming. Then, an important topic of future research is to investigate the efficiency of the presented techniques (through theoretical exploration or through experimental results). This further work should allow us to improve our methods obtaining more efficient algorithms.

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