1 Asymptotic Integration by Parts Formula and Regularity of Probability laws.

Abstract

We consider a sequence of random variables $F_n \sim p_n(x)dx$ which converge to a random variable F. If we know that $p_n \to p$ in some sweated sense, then we obtain $F \sim p(x)dx$. But in many interesting situations p_n blows up as $n \to \infty$. Our aim is to give a criterion which says that, if there is a "good equilibrium" between $\|F - F_n\|_1 \to 0$ and $\|p_n\| \uparrow \infty$ then we are still able to obtain the absolute continuity of the law of F and to study the regularity of the density p. Moreover we get some upper bounds for p. The blow up of p_n is characterized in terms of integration by parts formulae.

We give two examples. The first one is about diffusion processes with Hölder coefficients. The second one concerns the solution $f_t(dv)$ of the two dimensional homogeneous Boltzmann equation. We prove that, under some conditions on the parameters of the equation, we have $f_t(dv) = f_t(v)dv$. The initial distribution $f_0(dv)$ is a general measure (except a Dirac mass) so our result says that a regularization effect is at work; moreover, if the initial distribution has exponential moments $\int e^{|v|^\lambda} f_0(dv) < \infty$, then we prove that $f_t(v) \leq Ct^{-\eta} e^{-|v|^{\lambda'}}$ for every $\lambda' < \lambda$. So we have exponential upper bounds in space and at most polynomial blow up in time.