Sequential Monte Carlo methods

Lecture 2 – Probabilistic modelling of dynamical systems

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Outline – Lecture 2

Aim: Explain how latent variables and Markov chains are used in probabilistic modelling of dynamical system.

Outline:

1. Latent variables
2. Linear Gaussian state space model (LG-SSM)
3. Nonlinear state space model
4. Nonlinear filtering problem and its conceptual solution

Latent variable model

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The idea of introducing latent variables into models is probably one of the most **powerful concepts** in probabilistic modelling.

Cost: Learning the model becomes (significantly) harder.

Markov chain

The Markov chain is a probabilistic model that is used for modelling a sequence of states $(X_0, X_1, \ldots, X_T)$.

Definition (Markov chain)

A stochastic process $\{X_t\}_{t \geq 0}$ is referred to as a Markov chain if, for every $k > 0$ and t,

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A Markov chain is completely specified by:

1. An initial value X_0 and
2. a transition model (kernel) $\kappa(x_{t+1} \,|\, x_t)$ describing the transition from state X_t to state X_{t+1}, according to $X_{t+1} \,|\, (X_t = x_t) \sim \kappa(x_{t+1} \,|\, x_t)$.

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The state acts as a memory containing all information there is to know about the process at this point in time.

Markov chain

Our two most important applications of Markov chains in this course are:

1. The Markov model is used in the state space model (SSM) where we can only observe the state indirectly via a measurement that is related to the state.
2. The Markov chain constitutes the basic ingredient in the Markov chain Monte Carlo (MCMC) methods.

State space models

State space model (SSM)

The state space model (SSM) is a **Markov** chain that makes use of a **latent** variable representation to describe a **dynamical** process.

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2. observed process $\{Y_t\}_{t\geq 1}$ modelling the measurements and their relationship to the unobserved state process.

Linear Gaussian state space model

The linear Gaussian state space model (LG-SSM) is a model of a **dynamical system** where:

1. The dynamics are modeled using a Markov chain, describing the evolution of the **latent state** of the system,

$$X_t = AX_{t-1} + V_t, \qquad V_t \sim \mathcal{N}(0, Q).$$

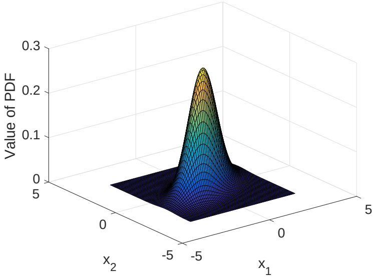
2. The measurements are modeled using

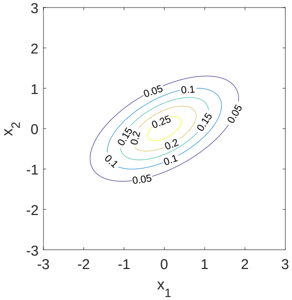
$$Y_t = CX_t + E_t, \qquad E_t \sim \mathcal{N}(0, R).$$

The Gaussian PDF

The PDF of a Gaussian variable is denoted $\mathcal{N}(x \mid \mu, \Sigma)$, i.e.,

$$\mathcal{N}(x \mid \mu, \Sigma) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma}} \exp\left(-\frac{1}{2}(x-\mu)^{\mathsf{T}}\Sigma^{-1}(x-\mu)\right)$$





Transition and observation density

Recall equations defining the LG-SSM,

1. **Transition model:** $X_t = AX_{t-1} + V_t, \quad V_t \sim \mathcal{N}(0, Q)$.
2. **Observation model:** $Y_t = CX_t + E_t, \quad E_t \sim \mathcal{N}(0, R)$.

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\begin{aligned}
X_t &= f(X_{t-1}, \theta) + V_t, \\
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where $\theta \in \mathbb{R}^{n_\theta}$ denotes static model parameters.

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The SSM offers a practical representation not only for modelling, but also for reasoning and inference.

Ex) “what are X_t, θ and Y_t”?

Aim (motion capture): Compute X_t (position and orientation of the different body segments) of a person (θ describes the body shape) moving around indoors using measurements Y_t (accelerometers, gyroscopes and ultrawideband).

Manon Kok, Jeroen D. Hol and Thomas B. Schön. **Using inertial sensors for position and orientation estimation.** *Foundations and Trends of Signal Processing*, 11(1-2):1-153, 2017.

Representing the SSM using distributions

Representation using probability distributions

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\begin{aligned}
X_t \mid (X_{t-1} = x_{t-1}, \theta) &\sim p(x_t \mid x_{t-1}, \theta), \\
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The unknown parameters can be modelled as either

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State inference: Learn about the state from the observations.
Parameter inference: Learn the (static) parameters from the observations.

SSM – full probabilistic model

The full probabilistic model is given by

$$p(\theta, x_{0:T}, y_{1:T}) = p(y_{1:T} \mid x_{0:T}, \theta)\, p(x_{0:T} \mid \theta) p(\theta)$$

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Factorizing the state prior:

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Distribution describing a parametric nonlinear SSM

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Model = probability distribution!

Nonlinear filtering problem

State inference

State inference refers to the problem of learning about the state $X_{k:l}$ based on the available measurements $Y_{1:t} = y_{1:t}$.

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Name	Probability density function
Filtering	$\mathbf{p}(\mathbf{x}_t \mid \mathbf{y}_{1:t})$
Joint filtering	$p(x_{0:t} \mid y_{1:t}), t = 1, 2, \ldots$
Prediction	$p(x_{t+1} \mid y_{1:t})$
Joint smoothing	$p(x_{1:T} \mid y_{1:T})$
Marginal smoothing	$p(x_t \mid y_{1:T}), t \leq T$

The nonlinear filtering problem

State filtering problem: Learn about the current state X_t based on the available measurements $Y_{1:t} = y_{1:t}$ when

$$\begin{aligned} X_t \mid (X_{t-1} = x_{t-1}) &\sim p(x_t \mid x_{t-1}), \\ Y_t \mid (X_t = x_t) &\sim p(y_t \mid x_t), \\ X_0 &\sim p(x_0). \end{aligned}$$

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$$X_0 \sim p(x_0). \qquad X_0 \sim p(x_0).$$

Goal: Compute the filter PDF $p(x_t \mid y_{1:t})$ as accurately as possible.

Filtering problem – conceptual solution

The measurement update

$$p(x_t \mid y_{1:t}) = \frac{\overbrace{p(y_t \mid x_t)}^{\text{measurement}} \; \overbrace{p(x_t \mid y_{1:t-1})}^{\text{prediction pdf}}}{p(y_t \mid y_{1:t-1})},$$

and the time update

$$p(x_t \mid y_{1:t-1}) = \int \underbrace{p(x_t \mid x_{t-1})}_{\text{dynamics}} \underbrace{p(x_{t-1} \mid y_{1:t-1})}_{\text{filtering pdf}} \, \mathrm{d}x_{t-1}.$$

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No closed-form solutions available except for a few special cases.

Explicit filtering solution for LG-SSM – Kalman filter

Linear transformations of Gaussian r.v. remain Gaussian and hence completely characterized by their mean and covariance.

Measurement update

$$
\begin{aligned}
p(x_t \mid y_{1:t}) &= \mathcal{N}\left(x_t \,\middle|\, \widehat{x}_{t\,|\,t}, P_{t|t}\right), \\
\widehat{x}_{t|t} &= \widehat{x}_{t|t-1} + K_t \left(y_t - C\widehat{x}_{t|t-1} - Du_t\right), \\
P_{t|t} &= (I - K_t C)P_{t|t-1}, \\
K_t &= P_{t|t-1}C^{\mathsf{T}} \left(CP_{t|t-1}C^{\mathsf{T}} + R\right)^{-1}.
\end{aligned}
$$

Time update

$$
\begin{aligned}
p(x_{t+1} \mid y_{1:t}) &= \mathcal{N}\left(x_{t+1} \,\middle|\, \widehat{x}_{t+1|t}, P_{t+1|t}\right), \\
\widehat{x}_{t+1|t} &= A\widehat{x}_{t|t} + Bu_t, \\
P_{t+1|t} &= AP_{t|t}A^{\mathsf{T}} + Q.
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Backward computations – (too) brief

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Marginal smoothing PDF

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Fredrik Lindsten and Thomas B. Schön. **Backward simulation methods for Monte Carlo statistical inference.** *Foundations and Trends in Machine Learning*, 6(1):1-143, 2013.

Beyond state space models

The nonlinear SSM is just a special case...

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1. a set of **vertices** $\mathcal{V}$ (nodes) represents the random variables
2. a set of **edges** $\mathcal{E}$ containing elements $(i, j) \in \mathcal{E}$ connecting a pair of nodes $(i, j) \in \mathcal{V}$
3. The arrows pointing to a certain node encodes which variables the corresponding node are conditioned upon.

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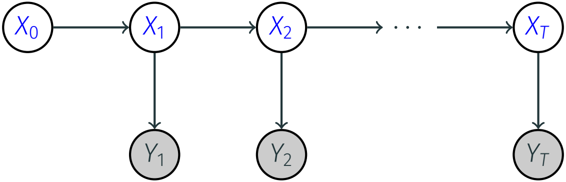

Christian A. Naesseth, Fredrik Lindsten and Thomas B. Schön. **Sequential Monte Carlo methods for graphical models.** *Advances in Neural Information Processing Systems (NIPS)*, Montreal, Quebec, Canada, December, 2014.

The nonlinear SSM is just a special case...

Representation using probabilistic program

`x[1] ? Gaussian(0.0, 1.0);`	$p(x_1)$
`y[1] ? Gaussian(x[1], 1.0);`	$p(y_1 \mid x_1)$
`for (t in 2..T) {`	
`  x[t] ? Gaussian(a*x[t - 1], 1.0);`	$p(x_t \mid x_{t-1})$
`  y[t] ? Gaussian(x[t], 1.0);`	$p(y_t \mid x_t)$
`}`	

A probabilistic program encodes a probabilistic model (here an LG-SSM) according to the semantics of a particular probabilistic programming language (here Birch).

Lawrence Murray and Thomas B. Schön. **Automated learning with a probabilistic programming language: Birch.** *Annual Reviews in Control*, 46:29-43, 2018.

Outlook – Gaussian process SSM

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The model functions f and g are assumed to be realizations from Gaussian process priors and $V_t \sim \mathcal{N}(0, Q)$, $E_t \sim \mathcal{N}(0, R)$.

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Task: Compute the posterior $p(f, g, Q, R, \eta, x_{0:T} \mid y_{1:T})$.

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Roger Frigola, Fredrik Lindsten, Thomas B. Schön, and Carl Rasmussen. **Bayesian inference and learning in Gaussian process state-space models with particle MCMC.** *NIPS*, 2013.

A few concepts to summarize lecture 2

Latent variable model: A model containing unknown variables that are not directly observed.

Markov chain: Described by an initial state and a transition kernel describing the transition from the present state to the next.

Spate space model (SSM): A latent variable model, where the latent variable (the state) is observed indirectly.

State inference: Learn about the state $X_{k:l}$ based on the available measurements $Y_{1:t} = y_{1:t}$.

Parameter inference: Learn the (static) parameters θ based on the available measurements $y_{1:T} = \{y_1, y_2, \ldots, y_T\}$.

Filtering problem: Learn about the current state X_t based on the available measurements $Y_{1:t} = y_{1:t}$ by computing $p(x_t \mid y_{1:t})$.

Kalman filter: Explicit solution to the state filtering problem when the SSM is linear and Gaussian.