Type-Based Structural Analysis for Modular Systems of Equations

Linköping University, 19 June 2014

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Background (1)

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- Ideally we would like to know, as early as possible:
  - Is a system of equations solvable?
  - Does an individual equation system fragment “make sense”; i.e., could it be part of a solvable system?

Background (2)

Studying/enforcing aspects of the relation between equations and variables occurring in them can help identifying problems, e.g.:

- Bunus and Fritzon's work on debugging Modelica models [1]
- Broman, Nyström, and Fritzon's work on enforcing balance between variables and equations through the Modelica type system [2]
- Modelica's notion of balanced models

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However, (static) ***guarantees*** only possible under very limited circumstances.

Quick Review of the Problem (1)

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$$x + y + z = 0 \tag{1}$$

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$$x + y + z = 0 \tag{1}$$

  - Does not have a (unique) solution.
  - Could be part of a system that does have a (unique) solution.
- The same holds for:

$$\begin{aligned} x - y + z &= 1 \\ z &= 2 \end{aligned} \tag{2}$$

Quick Review of the Problem (2)

- Composing (1) and (2):

$$\begin{aligned} x + y + z &= 0 \\ x - y + z &= 1 \\ z &= 2 \end{aligned} \tag{3}$$

Does have a solution.

Quick Review of the Problem (2)

- Composing (1) and (2):

$$\begin{aligned} x + y + z &= 0 \\ x - y + z &= 1 \\ z &= 2 \end{aligned} \tag{3}$$

  Does have a solution.

- However, the following fragment is over-constrained:

$$\begin{aligned} x &= 1 \\ x &= 2 \end{aligned} \tag{4}$$

Approaches (1)

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- Broman et al. annotate the ***type*** of components with difference between number of equations and number of variables.
- Type-based approach, so modular and early detection.
- Can confirm (3) OK while (4) has a problem.
- However, would accept:

$$
\begin{aligned}
x &= 1 \\
x &= 2 \\
x + y + z &= 0
\end{aligned}
\tag{5}
$$

Approaches (2)

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- Modelica has adopted a simplified notion of balance:
  - Models are required to be locally balanced
  - ***Partial*** models are allowed to be unbalanced
- Early detection, but type-level information limited to partial model or not.

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- Finds more problems than simple balance checking.
- Would reject (5)
- Based on analysing a complete system, so late detection.
  (Intended to be a debugging tool.)

This Talk

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  - First-class equation system fragments
  - Equation systems that vary over time (structural dynamism).

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This Talk

- Type-based approaches inherently modular.
- Sensible to look for type-based solutions if we wish to support
  - First-class equation system fragments
  - Equation systems that vary over time (structural dynamism).
- How far beyond basic balance checking can we go with a type-based approach?
- We'll investigate two approaches:
  - Incidence Type
  - Structural Well-Formedness

Functional Hybrid Modelling

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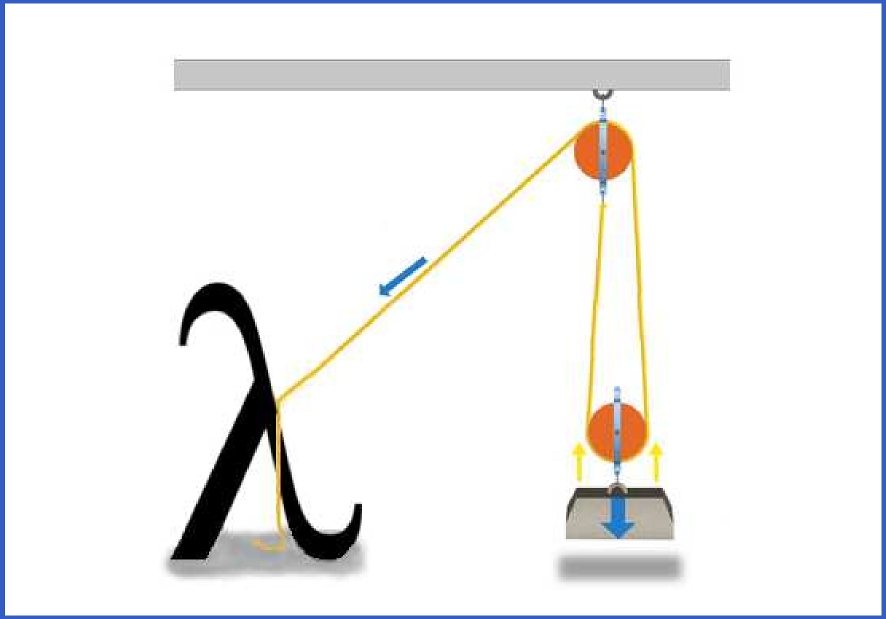
Our work has been carried out in the setting of Functional Hybrid Modelling (FHM), so let's opt for that.

(But remember: the ideas are generally applicable.)

The FHM Setting (1)

- FHM: functional approach to modelling and simulation of (physical) systems that can be described by an ***evolving*** set of differential equations.



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- Undirected equations: ***non-causal modelling***. (Differential Algebraic Equations, DAE; like Modelica)
- Two-level design:
  - ***equation level*** for modelling components
  - ***functional level*** for spatial and temporal composition of components
- Equations system fragments are frst-class entities at the functional level; viewed as relations on signals, or ***signal relations***.

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- Temporal composition: ***switching*** from one structural configuration into another.

The FHM Setting (4)

$$
\begin{aligned}
&\mathit{resistor} :: \texttt{Resistance} \rightarrow \texttt{SR}\,(\texttt{Pin}, \texttt{Pin})\\
&\mathit{resistor}\ r = \textbf{sigrel}\ (p, n)\ \textbf{where}\\
&\qquad\qquad \textbf{local}\ \mathsf{u}\\
&\qquad\qquad \mathit{twoPin} \diamond (p, n, u)\\
&\qquad\qquad r \cdot p.i = u
\end{aligned}
$$

The FHM Setting (4)

Parametrised model represented by ***function*** mapping parameters to a model. Note: first class models!

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Encapsulated equation system fragment.

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Encapsulated equation system fragment.

Signal relation ***application*** allows modular construction of models from component models.

The FHM Setting (5)

Composition by signal relation application (f_1, f_2 are known function symbols):

$$
\begin{array}{l}
foo :: SR\ (Real, Real, Real) \\
foo = \textbf{sigrel}\ (x_1, x_2, x_3)\ \textbf{where} \\
\quad \begin{array}{ll} f_1\ x_1\ x_2\ x_3 & = 0 \\ f_2\ x_2\ x_3 & = 0 \end{array}
\end{array}
$$

$$
\begin{array}{l}
foo \diamond (u, v, w) \\
foo \diamond (w, u + x, v + y)
\end{array}
$$

yields

$$
\begin{array}{ll}
f_1\ u\ v\ w & = 0 \\
f_2\ v\ w & = 0 \\
f_1\ w\ (u + x)\ (v + y) & = 0 \\
f_2\ (u + x)\ (v + y) & = 0
\end{array}
$$

The FHM Setting (7)

Evolving system of equations by switching blocks of equations in and out:

$$
\begin{array}{l}
\mathit{initially}\ [\,;\mathbf{when}\ \mathit{condition}_1] \Rightarrow \\
\quad \mathit{equations}_1 \\
\mathbf{when}\ \mathit{condition}_2 \Rightarrow \\
\quad \mathit{equations}_2 \\
\ldots \\
\mathbf{when}\ \mathit{condition}_n \Rightarrow \\
\quad \mathit{equations}_n
\end{array}
$$

The FHM Setting (8)

FHM is thus characterised by ***iterative staging***:

1. Compute model (“flat” system of equations)
2. Simulate (solve)
3. Repeat

Structural Non-singularity (1)

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- Structural non-singularity is (generally) neither a necessary nor sufficient condition for solvability.
- However typical solvers are predicated on the system being structurally non-singular.
- Insisting on structural non-singularity thus makes sense and is not overly onerous.

Structural Non-singularity (2)

Structural singularities can be discovered by studying the ***incidence matrix***:

Equations | Incidence Matrix

$$
\begin{aligned}
f_1(x, y, z) &= 0 \\
f_2(z) &= 0 \\
f_3(z) &= 0
\end{aligned}
\qquad
\begin{array}{c}
\begin{array}{ccc} x & y & z \end{array} \\
\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

A Possible Refinement (3)

So maybe we can index signal relations by incidence matrices?

$$foo :: SR\ (Real, Real, Real)\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{array}{l} foo = \mathbf{sigrel}\ (x_1, x_2, x_3)\ \mathbf{where} \\ \quad f_1\ x_1\ x_2\ x_3 = 0 \\ \quad f_2\ x_2\ x_3 \quad\ \ = 0 \end{array}$$

Incidence Type

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  - Incidence type of a ***system of equations***
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- The ***Incidence Type*** represents information about which variables occur in which equations.
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- Two interrelated instances:
  - Incidence type of a ***system of equations***
  - Incidence type of a ***signal relation***
- The incidence type of signal relation is obtained by ***abstraction*** over the incidence type of a system of equations as some variables are ***local***.

Composition of Incidence Types (1)

Recall

$$foo :: SR\ (Real, Real, Real)\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Consider

$$foo \diamond (u, v, w)$$
$$foo \diamond (w, u + x, v + y)$$

in a context with five variables u, v, w, x, y.

Composition of Incidence Types (2)

The incidence type for the equations obtained by instantiating foo is simply obtained by Boolean matrix multiplication. For $foo \diamond (u, v, w)$:

$$
\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}
\begin{array}{c}
\begin{array}{ccccc} u & v & w & x & y \end{array} \\
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}
\end{array}
=
$$

$$
\begin{array}{c}
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\end{array}
$$

Composition of Incidence Types (3)

For $foo \diamond (w, u + x, v + y)$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{ccccc} u & v & w & x & y \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} =$$

$$\begin{array}{c} \begin{array}{ccccc} u & v & w & x & y \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Composition of Incidence Types (4)

Complete incidence matrix and corresponding equations:

$$
\begin{array}{c}
\begin{array}{ccccc} u & v & w & x & y \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1
\end{pmatrix}
\end{array}
\qquad
\begin{aligned}
f_1\ u\ v\ w &= 0 \\
f_2\ v\ w &= 0 \\
f_1\ w\ (u+x)\ (v+y) &= 0 \\
f_2\ (u+x)\ (v+y) &= 0
\end{aligned}
$$

Abstraction over Incidence Types (1)

Now consider encapsulating the equations:

$$
\begin{aligned}
&bar = \mathbf{sigrel}\ (u, y)\ \mathbf{where} \\
&\quad \mathbf{local}\ v, w, x \\
&\quad foo \diamond (u, v, w) \\
&\quad foo \diamond (w, u + x, v + y)
\end{aligned}
$$

The equations of bar needs to be partitioned into:

- ***Local Equations***: equations used to (notionally) solve for the local variables
- ***Interface Equations***: equations contributed to the outside

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- ***A priori local equations***: equations over local variables only.
- ***A priori interface equations***: equations over interface variables only.
- ***Mixed equations***: equations over local and interface variables.

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How to partition?

- ***A priori local equations***: equations over local variables only.
- ***A priori interface equations***: equations over interface variables only.
- ***Mixed equations***: equations over local and interface variables.

Note: too few or too many local equations, or too many interface equations, means ***locally underdetermined*** or ***overdeteremined*** systems of equations.

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In our case:

- We have 1 a priori local equation, 3 mixed equations
- We need to choose 3 local equations (as 3 local variables) and 1 interface equation
- Consequently, ***3*** possibilities, yielding the following possible incidence types for bar:

$$\begin{array}{cc} u & y \\ \end{array} \qquad \begin{array}{cc} u & y \\ \end{array} \qquad \begin{array}{cc} u & y \\ \end{array}$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}$$

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Details in [1].

Pros and Cons

Works for analysis in a first-order setting.
However:

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- The matrix notation is cumbersome.
- Type annotations would often be needed.
- Type-checking seems expensive.

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Is there a middle ground between incidence types and basic equation-variable balance?

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Structural Well-Formedness (1)

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- a better approximation of structural non-singularity than equation-variable balance;
- less precise, but much more practical (for FHM at least) than incidence types.

Signal relation types are indexed by ***balance***: henceforth meaning the ***number of contributed equations***. Further, ***constraints*** on balance variables:

$$(2 \leq m \leq 4, 3 \leq n \leq 5) \Rightarrow SR\ (\ldots)\ m \to SR\ (\ldots)\ n$$

Structural Well-Formedness (2)

The approach distinguishes two kinds of variables:

- ***interface variables*** (i_Z)
- ***local variables*** (l_Z)

and three kinds of equations:

- ***interface equations*** (i_Q)
- ***mixed equations*** (m_Q)
- ***local equations*** (l_Q)

Total number of equations: $a_Q = i_Q + m_Q + l_Q$.

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Total number of equations: $a_Q = i_Q + m_Q + l_Q$.
Note: abstraction of earlier notion of local equation etc.

Structural Well-Formedness (3)

A signal relation is ***structurally well-formed*** (SWF) iff:

1. $l_Q + m_Q \geq l_Z$
2. $l_Q \leq l_Z$
3. $i_Q \leq i_Z$
4. $a_Q - l_Z \leq i_Z$
5. $i_Q \geq 0$, $m_Q \geq 0$, $l_Q \geq 0$

The balance (contribution) of a SWF relation is $n = a_Q - l_Z$.

Structural Dynamism

Recall that FHM allows for an evolving system of equations:

$$
\begin{array}{l}
\mathit{initially}\ [\,;\mathbf{when}\ \mathit{condition}_1\,] \Rightarrow \\
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What about structural well-formedness?

Structural Dynamism and SWF (1)

Exactly one switch-branch active at any point. How should the number of equations in each branch be related?

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- ***Strong Approach***: exactly the same number of interface, mixed, and local equations in each branch.
  Very restrictive.
- ***Weak Approach***: same number of equations in each branch.
  Loses too much information.
- ***Fair Approach***: branches are reconcilable.

Structural Dynamism and SWF (2)

A switch-block is ***reconcilable***, contributing i interface equations, m mixed equations, l local equations, iff i, m, l satisfying the following constraints for each branch k can be found:

6. $i \geq i_k \geq 0$
7. $l \geq l_k \geq 0$
8. $m \leq m_k - (i - i_k) - (l - l_k)$
9. $i + m + l = i_k + m_k + l_k$

Note: Interestingly, m may be negative!

Preservation? (1)

Consider:

$$
\begin{array}{l}
foo = \textbf{sigrel}\ (x, y)\ \textbf{where} \\
\quad \textbf{local}\ z \\
\quad f\ (x, y, z) = 0 \\
\quad g\ (x) = 0 \\
fie = \textbf{sigrel}\ (u)\ \textbf{where} \\
\quad \textbf{local}\ v \\
\quad foo \diamond (u, v)
\end{array}
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&\mathit{fie} = \textbf{sigrel}\ (u)\ \textbf{where}\\
&\quad \textbf{local}\ v\\
&\quad \mathit{foo} \diamond (u, v)
\end{aligned}
$$

Both foo and fie are structurally well-formed (why?) with balance 1 and 0, respectively.

Preservation? (2)

But if we carry out some “flattening”:

$$
\begin{aligned}
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- Similarities to e.g. Hybrid Types [Knowles and Flanagan].

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