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Second-Order Quantifier Elimination in Higher-Order Contexts with Applications to the Semantical Analysis of Conditionals

Abstract. Second-order quantifier elimination in the context of classical logic emerged as a powerful technique in many applications, including the correspondence theory, relational databases, deductive and knowledge databases, knowledge representation, commonsense reasoning and approximate reasoning. In the current paper we first generalize the result of Nonnengart and Szałas [17] by allowing second-order variables to appear within higher-order contexts. Then we focus on a semantical analysis of conditionals, using the introduced technique and Gabbay's semantics provided in [10] and substantially using a third-order accessibility relation. The analysis is done via finding correspondences between axioms involving conditionals and properties of the underlying third-order relation.

Keywords: conditionals, second-order quantifier elimination, higher-order relations.

1. Introduction

Second-order quantifier elimination in the context of classical logic emerged as a powerful technique in many applications, including the correspondence theory [9, 16, 17, 19, 22, 23, 24, 25], relational databases [8, 13], deductive and knowledge databases [4], knowledge representation, commonsense reasoning and approximate reasoning [5, 6, 7, 15] (for a comprehensive overview of the area see [11]). All of the quoted results are based on elimination of predicate variables from formulas of the classical second-order logic. On the other hand, some important semantical phenomena have their natural counterparts in higher-order logics. It is then desirable to provide tools dealing directly with higher-order contexts.

In the current paper we first generalize the result of Nonnengart and Szałas [17] by allowing second-order variables to appear within higher-order contexts. Up to now the only technique allowing one to deal with higher-order contexts has been considered in [18]. However, the considerations of [18] have been restricted to the elementary set theory. In the current paper we work with higher-order relations assuming their standard semantics only.

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As a case study showing usefulness of the introduced technique we consider the semantics of conditionals. Conditionals have been studied by many authors (see, e.g., [2, 10, 14, 21]). In [10] Gabbay has argued that conditionals require the use of a third-order relation. Namely, a conditional statement $\alpha > \beta$ asserts that β follows from α under "certain" conditions, which depend on the meaning of α and β and on the properties of the world in which $\alpha > \beta$ was uttered. This leads to the following semantics of $\alpha > \beta$ (for a closer explanation see [10] and Section 3.2 below):

$$s \models \alpha > \beta \text{ iff for all } t \text{ such that } R(\alpha(u), \beta(v), s, t), \text{ we have } t \models \alpha \to \beta,$$

where s, t are worlds and R is an accessibility relation. Using this semantics we can find correspondences between axioms involving conditionals and properties of R. For our purposes an adaptation of the modal correspondence theory (see, e.g., [24, 25]) is suitable. In fact, we shall apply the method based on elimination of second-order quantifiers, as done, e.g., in [9, 22, 17, 3].

To illustrate the technique consider the axiom $(\text{TRUE} > \alpha) \to \alpha$. According to Gabbay's semantics, this axiom is equivalent to

$$\forall A \forall x [\forall y (R(\text{TRUE}, A(v), x, y) \to A(y)) \to A(x)].$$

The third-order context is obvious here. For a more comprehensive treatment of this axiom see Example 3.5.

The paper is structured as follows. Section 2 provides a generalization of the Ackermann's lemma of [1] and the fixpoint theorem of Nonnengart and Szałas [17] to the case of higher-order contexts. Even if the Ackermann's lemma is a special case of the fixpoint theorem, we consider it separately, as it does not make use of fixpoints. In Section 3 we recall the language of conditionals together with Gabbay's semantics and show applications of the introduced technique to the semantical analysis of conditionals. Finally, Section 4 concludes the paper.

2. Second-Order Quantifier Elimination in Higher-Order Contexts

Let us first introduce the notion of higher-order relations, where 2^U denotes the set of all subsets of a set U,

$$\mathcal{P}^n(U) \stackrel{\text{def}}{=} \begin{cases} U & \text{when } n = 1 \\ 2^{\mathcal{P}^{n-1}(U)} & \text{when } n > 1. \end{cases}$$

Definition 2.1. For $k \in \omega$ and $1 \leq n \in \omega$, k-argument *relations of order* n are defined to be subsets of $U_1 \times \ldots \times U_k$, where, for $1 \leq i \leq k$, $U_i = \mathcal{P}^{a_i}(U)$, for some $1 \leq a_i \leq n$. *Higher order relations over a set* U are relations of order $n \geq 2$.

Higher-order relation symbols denote higher-order relations. *Higher-order formulas* are defined by extending the definition of the classical second-order logic by assuming that higher-order relation symbols can occur wherever the classical first-order relation symbols can.

We shall say that a relation S is *compatible* with relation X iff S and X have the same arities and respective arguments are of the same order. $\triangleleft$

Definition 2.2. By a *model* we understand a pair $\mathcal{M} \stackrel{\text{def}}{=} \langle \mathcal{I}, \mathcal{V} \rangle$, where $\mathcal{I}$ is the classical (first-order) relational structure and $\mathcal{V}$ is an assignment of domain elements to individual variables, relations to first-order variables and higher-order relations to compatible higher-order relation symbols. By $\alpha^{\mathcal{M}}$ we understand a higher-order relation of $\mathcal{M}$ which is the interpretation of α in $\mathcal{M}$.

Let $\mathcal{M}$ be a model. We shall say that a formula $\alpha(X)$ is *up-monotone* (respectively *down-monotone*) *w.r.t. a relation symbol* X in $\mathcal{M}$ iff for all relations R, S of $\mathcal{M}$ compatible with X, if $R \subseteq S$ then $\alpha^{\mathcal{M}}(R) \subseteq \alpha^{\mathcal{M}}(S)$ (respectively, $\alpha^{\mathcal{M}}(S) \subseteq \alpha^{\mathcal{M}}(R)$). $\triangleleft$

Let $\alpha(x, \bar{y})$ be a higher-order formula, $X(\bar{x})$ be a (higher-order) relation, $\gamma(\bar{x})$ be a (higher-order) formula with all free variables being $\bar{x}$. Then $\alpha^{X(\bar{x})}_{\gamma(\bar{x})}$ denotes the formula obtained from α by substituting all subformulas of the form $X(\bar{t})$ by $\gamma(\bar{t})$.

The following Ackermann-like lemma allows us to deal with relations of arbitrary order.

Lemma 2.3. Let X be a predicate variable and $\alpha(\bar{x}, \bar{z})$, $\beta(X)$ be formulas with relations of arbitrary order, where the number of distinct variables in $\bar{x}$ is equal to the arity of X. Let α contain no occurrences of X.
If $\beta(X)$ is up-monotone w.r.t. X then

$$\exists X \big\{ \forall \bar{x} [X(\bar{x}) \rightarrow \alpha(\bar{x}, \bar{z})] \wedge \beta(X) \big\} \;\equiv\; \beta(X)^{X(\bar{x})}_{\alpha(\bar{x}, \bar{z})}. \tag{1}$$

If $\beta(X)$ is down-monotone w.r.t. X then

$$\exists X \big\{ \forall \bar{x} [\alpha(\bar{x}, \bar{z}) \rightarrow X(\bar{x})] \wedge \beta(X) \big\} \;\equiv\; \beta(X)^{X(\bar{x})}_{\alpha(\bar{x}, \bar{z})}, \tag{2}$$

where variables of $\bar{z}$ in formulas $\alpha(\bar{x}, \bar{z})$ of equivalences (1) and (2) are treated as parameters. $\triangleleft$

The above lemma does not make use of fixpoints. In the following theorem, extending the fixpoint theorem of Nonnengart and Szałas [17], by $[\mathrm{LFP}\, X(\bar{x}).\alpha(X,\bar{x},\bar{z})]$ and $[\mathrm{GFP}\, X(\bar{x}).\alpha(X,\bar{x},\bar{z})]$ we denote the least and the greatest fixpoint of $\alpha(X,\bar{x},\bar{z})$, i.e., the least and the greatest (w.r.t. inclusion) relation satisfying $X(\bar{x}) \equiv \alpha(X,\bar{x},\bar{z})$. We shall only use this notation in contexts where the least and the greatest relation exist.

The following theorem generalizes the one given in [17].

THEOREM 2.1. *Let X be a predicate variable and $\alpha(X,\bar{x},\bar{z})$, $\beta(X)$ be formulas with relations of arbitrary order, where the number of distinct variables in $\bar{x}$ is equal to the arity of X. Let α be up-monotone w.r.t. X.*
If $\beta(X)$ is up-monotone w.r.t. X then

$$\exists X\big\{\forall\bar{x}[X(\bar{x}) \to \alpha(X,\bar{x},\bar{z})] \land \beta(X)\big\} \;\equiv\; \beta(X)^{X(\bar{x})}_{[\mathrm{GFP}\, X(\bar{x}).\alpha(X,\bar{x},\bar{z})](\bar{x})}. \tag{3}$$

If $\beta(X)$ is down-monotone w.r.t. X then

$$\exists X\big\{\forall\bar{x}[\alpha(X,\bar{x},\bar{z}) \to X(\bar{x})] \land \beta(X)\big\} \;\equiv\; \beta(X)^{X(\bar{x})}_{[\mathrm{LFP}\, X(\bar{x}).\alpha(X,\bar{x},\bar{z})](\bar{x})}. \tag{4}$$

PROOF. We prove equivalence (3). A proof of (4) can be carried out similarly. Let $\mathcal{M} = \langle \mathcal{I}, \mathcal{V}\rangle$ be a model.

$(\to)$
Assume that $\mathcal{M} \models \exists X\big\{\forall\bar{x}[X(\bar{x}) \to \alpha(X,\bar{x},\bar{z})] \land \beta(X)\big\}$. Thus, there is $\mathcal{V}'$ extending $\mathcal{V}$ to cover X, such that

$$\langle I, \mathcal{V}'\rangle \models \forall\bar{x}[X(\bar{x}) \to \alpha(X,\bar{x},\bar{z})] \land \beta(X),$$

from which we obtain $\langle I, \mathcal{V}'\rangle \models \forall\bar{x}[X(\bar{x}) \to \alpha(X,\bar{x},\bar{z})]$. Note that, by assumption, $\alpha(X,\bar{x},\bar{z})$ is up-monotone w.r.t. X. Therefore we also have that $\langle I, \mathcal{V}'\rangle \models \forall\bar{x}\big[X(\bar{x}) \to [\mathrm{GFP}\, X(\bar{x}).\alpha(X,\bar{x},\bar{z})]\big]$.

Since, by assumption, $\beta(X)$ is up-monotone w.r.t. X, we have

$$\langle I, \mathcal{V}'\rangle \models \beta(X)^{X(\bar{x})}_{[\mathrm{GFP}\, X(\bar{x}).\alpha(X,\bar{x},\bar{z})](\bar{x})}.$$

Observe now that formula $\beta(X)^{X(\bar{x})}_{[\mathrm{GFP}\, X(\bar{x}).\alpha(\bar{x},\bar{z})](\bar{x})}$ does not contain free occurrences of X (all such occurrences are bound by the fixpoint operator), which implies that $\mathcal{V}$ and $\mathcal{V}'$ are equal on its variables. In consequence,

$$\mathcal{M} \models \beta(X)^{X(\bar{x})}_{[\mathrm{GFP}\, X(\bar{x}).\alpha(X,\bar{x},\bar{z})](\bar{x})}.$$

(←)
Assume that $\mathcal{M} \models \beta(X)^{X(\bar{x})}_{[\text{GFP}\, X(\bar{x}).\alpha(X,\bar{x},\bar{z})](\bar{x})}$. Define

$$X(\bar{x}) \stackrel{\text{def}}{\equiv} [\text{GFP}\, X(\bar{x}).\alpha(X, \bar{x}, \bar{z})](\bar{x}).$$

Then, since X is a fixpoint of $\alpha(X, \bar{x}, \bar{z})$, we have that

$$\mathcal{M} \models \forall \bar{x}[X(\bar{x}) \equiv \alpha(X, \bar{x}, \bar{z})],$$

from which $\mathcal{M} \models \forall \bar{x}[X(\bar{x}) \to \alpha(X, \bar{x}, \bar{z})]$.

By assumption, $\mathcal{M} \models \beta(X)$ holds. Thus we have exhibited X such that $\mathcal{M} \models \forall \bar{x}[X(\bar{x}) \to \alpha(\bar{x}, \bar{z})] \wedge \beta(X)$, i.e., we have that

$$\mathcal{M} \models \exists X \big\{ \forall \bar{x}[X(\bar{x}) \to \alpha(\bar{x}, \bar{z})] \wedge \beta(X) \big\}.$$ ■

Remark 2.4. Observe that Theorem 2.1 subsumes Lemma 2.3. Namely if the formula $\alpha(X, \bar{x}, \bar{z})$ in (3) and (4) does, in fact, not contain X then

$$[\text{GFP}\, X(\bar{x}).\alpha(X, \bar{x}, \bar{z})] \equiv [\text{LFP}\, X(\bar{x}).\alpha(X, \bar{x}, \bar{z})] \equiv \alpha(X, \bar{x}, \bar{z})$$

and Theorem 2.1 reduces to Lemma 2.3. We consider Lemma 2.3 separately, as it simplifies the results. ◁

3. Conditionals

3.1. Introduction

A conditional is an expressions of the form "if then" (see, e.g., [2, 10, 14, 21]). There are various kinds of conditionals that fit into that pattern, such as *counterfactual conditionals* "if it were the case that α then it would be the case that β" with α being False in the actual world, *causal conditionals* "if α then causally β", etc. What is common to all these constructions is that the antecedent is connected to the consequent in such a way that the antecedent represents a condition or a context for the consequent or vice versa.

Beginning with work of Stalnaker and Lewis [20, 14] several formal treatments of conditionals have been proposed. Most of them are based on the notion of similarity between possible worlds. Basically, a conditional $\alpha > \beta$ is True in a world s if and only if β is true in every α-world most similar to s. Of course, there are many choices as to the properties of similarity between worlds (for the discussion of possible choices see, e.g., [12]). For example, the *Burgess-Lewis semantics* (see, [2]) is given by assuming a set W of worlds and a ternary relation $R(s, t, u)$ on W with the intuitive meaning that t is more similar to s than u. A further requirement is that for all

$s \in W$, a binary relation $R_s(t, u)$ obtained from R by fixing s, is irreflexive and transitive.

In [10] Gabbay has argued that conditionals often require an approach which differs from those quoted above. In his approach a conditional statement $\alpha > \beta$ asserts that β follows from α under "certain" conditions, which depend on the meaning of α and β and on the properties of the world in which $\alpha > \beta$ was uttered. For example, saying "if I were the president, I would have withdrawn from the East" one means that, the political situation being the same, β follows from α. So in order to falsify that statement, one has to present a possible world where both the general political situation is the same and I am president but where I do not withdraw from the East.

Generally, whenever a statement $\alpha > \beta$ is uttered at a world s, the speaker has in mind a certain set of statements $\Delta(\alpha, \beta, s)$ which is supposed to remain true and the speaker wants to express that in all worlds validating Δ, formula $\alpha \to \beta$ must hold. The set Δ depends both on α and β, for consider the statements:

1. if New York were in Georgia, then New York would be in the South
2. if New York were in Georgia, then Georgia would be in the North.

Clearly, in the first sentence "Georgia is in the South" must retain its truth value and in the second "New York is in the North" must retain its truth value.

In the next sections we shall discuss semantical issues of conditionals and show applications of second-order quantifier elimination techniques introduced in Section 2. Herzig [12] has been the first author who applied such techniques (in fact, the Scan algorithm of [9]) to the analysis of conditionals. Here, however, we deal with third-order relations which are out of scope of Scan.

3.2. Language and Semantics

The language of conditionals assumes a basic logic, say $\mathcal{L}$, and extends it's syntax by assuming that $\alpha > \beta$ is a formula, if α and β are. In what follows we shall assume that $\mathcal{L}$ is the classical propositional calculus with V_0 as the set of propositional variables and $\{\neg, \vee, \wedge, \to, \equiv\}$ as the set of propositional connectives.

The truth value of $\alpha > \beta$, proposed in [10], is the following:

> $\alpha > \beta$ is True at s iff in all possible worlds in which $\Delta(\alpha, \beta, s)$ and α are True, β is also True.

If we take $\Box$ to mean the modal necessity we get:

$$\alpha > \beta \text{ is } \textsc{True} \text{ at } s \text{ iff } \Box\Big(\bigwedge_{\delta\in\Delta(\alpha,\beta,s)} \delta \rightarrow (\alpha \rightarrow \beta)\Big).$$

In order to avoid possibly infinite conjunctions, one can use a (third-order) relation $R(X, Y, x, y)$, where X, Y are sets of worlds and x, y are worlds,[1] and define

$$\alpha > \beta \text{ is } \textsc{True} \text{ at } s \text{ iff } \alpha \rightarrow \beta \text{ is true in all worlds } t, \text{ where } R(\alpha(u), \beta(v), s, t) \text{ holds,}$$

where u and v are fresh free variables. We understand $\alpha(u)$ and $\beta(v)$ as characteristic formulas for sets of worlds in which these formulas are true. For example, $\alpha(u)$ represents the set $\{u \mid \alpha(u) \text{ holds }\}$. In what follows we sometimes abuse notation and use set inclusion on formulas, identifying inclusion with the corresponding implication.

The semantics of conditionals can now be defined more precisely,

$$s \models \alpha > \beta \text{ iff for all } t \text{ such that } R(\alpha(u), \beta(v), s, t), \text{ we have } t \models \alpha \rightarrow \beta. \tag{5}$$

Finally, based on (5) we define a translation of formulas involving conditionals into the formulas of the classical first-order logic. This translation provides a precise semantics of conditionals.

Definition 3.1. The *translation* $T(\alpha, x)$, where α is a formula and x is a world, is defined recursively:

$$T(\alpha, x) \stackrel{\text{def}}{=} A(x), \text{where } \alpha \in V_0 \text{ is a propositional variable and } A \text{ is a unary relation symbol corresponding to } \alpha$$

$$T(\neg\alpha, x) \stackrel{\text{def}}{=} \neg T(\alpha, x)$$

$$T(\alpha \circ \beta, x) \stackrel{\text{def}}{=} T(\alpha, x) \circ T(\beta, x), \text{where } \circ \in \{\neg, \vee, \wedge, \rightarrow, \equiv\}$$

$$T(\alpha > \beta, x) \stackrel{\text{def}}{=} \forall y\big[R(A(u), B(v), x, y) \rightarrow (T(\alpha, y) \rightarrow T(\beta, y))\big], \text{ where } u, v \text{ are fresh variables.}$$

◁

In order to find correspondences between axioms involving conditionals and properties of the considered accessibility relation we consider third-order formulas of the form

$$\forall \bar{A} \forall x\big[T(\alpha, x)\big],$$

[1] The intuitive meaning of relation $R(X, Y, x, y)$ is that the world y is accessible from the world x relative to X and Y.

where α is the considered axiom and $\bar{A}$ is a sequence consisting of all relation symbols corresponding to propositions in A and introduced by the translation T.

3.3. Applications of Second-Order Quantifier Elimination Techniques

In order to apply Lemma 2.3 and Theorem 2.1 we need to know the monotonicity properties of $R(X, Y, s, t)$, where R is the accessibility relation required in (5).

We can first observe that whenever A implies A' (in all worlds) then

$$(A' > B) \to (A > B)$$

should also hold, meaning that

$$\forall y\big[R(A'(u), B(v), x, y) \to \big(A'(y) \to B(y)\big)\big] \to \\ \forall y\big[R(A(u), B(v), x, y) \to \big(A(y) \to B(y)\big)\big].$$

Therefore, in order to make sure that $A(y) \to B(y)$ will hold, we would like to use the transitivity of implication and the facts that $A(y) \to A'(y)$ and $A'(y) \to B(y)$. But the latter fact is guaranteed for those y that are accessible from x via $R(A'(u), B(v), x, y)$. Thus

$$R(A(u), B(v), x, y) \subseteq R(A'(u), B(v), x, y)$$

guarantees the desired property. We will then assume that

$$A \subseteq A' \text{ implies } R(A(u), B(v), x, y) \subseteq R(A'(u), B(v), x, y), \tag{6}$$

which is up-monotonicity of R w.r.t. its first coordinate.

Also, whenever B implies B' (i.e., $B \subseteq B'$) then $(A > B) \to (A > B')$ should also hold, which means that

$$\forall y\big[R(A(u), B(v), x, y) \to \big(A(y) \to B(y)\big)\big] \to \\ \forall y\big[R(A(u), B'(v), x, y) \to \big(A(y) \to B'(y)\big)\big].$$

Similarly to the previous case, we can notice that

$$R(A(u), B'(v), x, y) \subseteq R(A(u), B(v), x, y)$$

guarantees that this formula indeed holds. We will then assume that

$$B \subseteq B' \text{ implies } R(A(u), B'(v), x, y) \subseteq R(A(u), B(v), x, y), \tag{7}$$

which is down-monotonicity of R w.r.t. its second coordinate.

Summarizing (6) and (7), we further on assume that R is up-monotone w.r.t. its first argument and down-monotone w.r.t. its second argument. In particular we make these assumptions in Examples 3.2– 3.6 below.

Example 3.2. Consider the axiom

$$\alpha > \alpha. \tag{8}$$

According to the semantics given by (5), axiom (8) is equivalent to

$$\forall A \forall x [\forall y (R(A(u), A(v), x, y) \to (A(y) \to A(y)))].$$

Since the $A(y) \to A(y)$ is a tautology, the above formula and therefore also axiom (8), reduces to True. ◁

Example 3.3. Consider the axiom

$$(\alpha \wedge \beta) \to (\alpha > \beta). \tag{9}$$

According to the semantics given by (5), axiom (9) is equivalent to

$$\begin{split}\forall A \forall B \forall x \Big[\big(A(x) \wedge B(x)\big) \to \forall y \big(R(A(u), B(v), x, y) \to \\ (A(y) \to B(y))\big)\Big].\end{split} \tag{10}$$

Formula (10) is equivalent to

$$\neg \exists A \exists B \exists x \Big[A(x) \wedge B(x) \wedge \exists y \big(R(A(u), B(v), x, y) \wedge A(y) \wedge \neg B(y)\big)\Big],$$

i.e., to

$$\neg \exists x \exists y \exists A \exists B \Big[A(x) \wedge B(x) \wedge R(A(u), B(v), x, y) \wedge A(y) \wedge \neg B(y)\Big], \tag{11}$$

According to our assumption (6), relation $R(A(u), B(v), x, y)$ is up-monotone w.r.t. A, therefore A can be replaced by True and we obtain the following formula equivalent to (11):

$$\neg \exists x \exists y \exists B \Big[R(\textsc{True}, B(v), x, y) \wedge B(x) \wedge \neg B(y)\Big]. \tag{12}$$

Now observe that (12) is equivalent to

$$\neg \exists x \exists y \exists B \Big[\forall z \big[x = z \to B(z)\big] \wedge R(\textsc{True}, B(v), x, y) \wedge \neg B(y)\Big]. \tag{13}$$

According to our assumption (7), formula

$$R(\textsc{True}, B(v), x, y) \wedge \neg B(y)$$

is down-monotone w.r.t. B. Applying Lemma 2.3 we then obtain the following equivalent of (13):

$$\neg\exists x\exists y\Big[R(\textsc{True}, x = v, x, y) \wedge y \neq x\Big],$$

which is equivalent to

$$\forall x\forall y\Big[R(\textsc{True}, x = v, x, y) \rightarrow y = x\Big].$$

Slightly abusing our notation, the above formula is equivalent to

$$\forall x\forall y\Big[R\big(\textsc{True}, \{x\}, x, y\big) \rightarrow y = x\Big].$$

Intuitively, our initial axiom (9) requires that the conjunction $A \wedge B$ implies the conditional $A > B$. This means that A should imply B in all worlds accessible from the current world via the relation $R(A(u), B(v), x, y)$, i.e., that

$$\forall y\big[R(A(u), B(v), x, y) \rightarrow (A(y) \rightarrow B(y))\big].$$

To see the connection we consider two cases.

If $y = x$ then $A(y) \rightarrow B(y)$ holds since $A \wedge B$ holds in the current world x which is the same as y.

If $x \neq y$ then indeed $R\big(\textsc{True}, \{x\}, x, y\big)$ should not hold for otherwise we could construct a model for $A \wedge B$ and $\neg(A > B)$, consisting of two worlds $\{x, y\}$ with x being the current world, and satisfying $A(x), A(y), B(x)$ and $\neg B(y)$. Now the set of worlds satisfying A would be $\{x, y\}$ and the set of worlds satisfying B would be $\{x\}$. Thus $R(A(u), B(v), x, y)$ is equivalent to $R\big(\{x, y\}, \{x\}, x, y\big)$. If this was true then y would be accessible from x and $A(y) \rightarrow B(y)$ should be True, which is not the case since $A(y)$ is True and $B(y)$ is False. ◁

Example 3.4. Consider the axiom

$$(\alpha > \textsc{True}) \rightarrow \neg\alpha. \tag{14}$$

According to the semantics given by (5), this formula is equivalent to

$$\forall A\forall x[\forall y(R(A(u), \textsc{True}, x, y) \rightarrow (A(y) \rightarrow \textsc{True})) \rightarrow \neg A(x)],$$

i.e., to $\forall A \forall x[\neg A(x)]$, which, even without eliminating $\forall A$, can be easily seen to be equivalent to FALSE. Therefore there is no frame satisfying axiom (14). On the other hand, (14) should be valid for counterfactuals. Therefore this example shows that, in some cases, counterfactuals cannot be adequately captured by the semantics we consider.

It would be interesting to check what formulas α would make the axiom (14) satisfiable in a frame. We are then interested in checking which formulas A satisfy in the current world x

$$\forall y(R(A(u), \text{TRUE}, x, y) \to (A(y) \to \text{TRUE})) \to \neg A(x), \tag{15}$$

and these are exactly those which satisfy $\neg A(x)$, since formula (15) is equivalent to $\neg A(x)$, which is what should have been expected. ◁

EXAMPLE 3.5. Consider the axiom

$$(\text{TRUE} > \alpha) \to \alpha. \tag{16}$$

According to the semantics given by (5), this formula is equivalent to

$$\forall A \forall x \Big[\forall y\big(R(\text{TRUE}, A(v), x, y) \to A(y)\big) \to A(x)\Big],$$

i.e., to

$$\neg \exists x \exists A \Big[\forall y\big(\neg R(\text{TRUE}, A(v), x, y) \lor A(y)\big) \land \neg A(x)\Big].$$

and further to

$$\neg \exists x \exists A \Big[\forall y\big(\neg R(\text{TRUE}, A(v), x, y) \lor A(y)\big) \land \forall z\big(A(z) \to x \neq z\big)\Big]. \tag{17}$$

Observe that due to the assumption (7), formula $\neg R(\text{TRUE}, A(v), x, y) \lor A(y)$ is up-monotone wrt A. Therefore we can apply Lemma 2.3 and obtain the following formula equivalent to (17):

$$\neg \exists x \forall y\big(\neg R(\text{TRUE}, x \neq v, x, y) \lor x \neq y\big). \tag{18}$$

Formula (18) is itself equivalent to

$$\forall x \exists y\big(R(\text{TRUE}, x \neq v, x, y) \land x = y\big),$$

i.e., to $\forall x\big(R(\text{TRUE}, x \neq v, x, x)\big)$ or, slightly abusing notation, to

$$\forall x\big(R(\text{TRUE}, -\{x\}, x, x)\big),$$

which is a form of reflexivity $R(A, B, x, x)$ for all A and all B such that $x \notin B$ (due to the down-monotonicity of R w.r.t. its arguments).

Since TRUE $>$ TRUE is a tautology, combining this fact with our result we have that $R(A, B, x, x)$ validating axiom (16) holds for any formulas A and B and any world x. ◁

EXAMPLE 3.6. Consider the axiom

$$\big((\alpha > \neg\alpha) \rightarrow \neg\alpha\big) \rightarrow (\alpha > \textsc{False}). \tag{19}$$

According to the semantics given by (5), this formula is equivalent to

$$\forall A \forall x \Big[\Big(\forall y \big(R(A(u), \neg A(v), x, y) \rightarrow (A(y) \rightarrow \neg A(y))\big) \rightarrow \neg A(x)\Big) \rightarrow \\ \forall z[R(A(w), \textsc{False}, x, z) \rightarrow (A(z) \rightarrow \textsc{False})]\Big],$$

i.e., to

$$\forall A \forall x \Big[\Big(\forall y \big(R(A(u), \neg A(v), x, y) \rightarrow \neg A(y)\big) \rightarrow \neg A(x)\Big) \rightarrow \\ \forall z[R(A(w), \textsc{False}, x, z) \rightarrow \neg A(z)]\Big],$$

and further to

$$\neg \exists x \exists A \Big[\Big(\forall y \big(R(A(u), \neg A(v), x, y) \rightarrow \neg A(y)\big) \rightarrow \neg A(x)\Big) \wedge \\ \exists z[R(A(w), \textsc{False}, x, z) \wedge A(z)]\Big],$$

and to

$$\neg \exists x \exists A \Big[\Big(A(x) \rightarrow \exists y \big(R(A(u), \neg A(v), x, y) \wedge A(y)\big)\Big) \wedge \\ \exists z[R(A(w), \textsc{False}, x, z) \wedge A(z)]\Big],$$

and finally to

$$\neg \exists x \exists A \Big[\forall t \Big(A(t) \rightarrow \big(x \neq t \vee \exists y \big(R(A(u), \neg A(v), x, y) \wedge A(y)\big)\big)\Big) \wedge \\ \exists z[R(A(w), \textsc{False}, x, z) \wedge A(z)]\Big].$$

By assumptions (6) and (6), $R(A(u), \neg A(v), x, y)$ is up-monotone w.r.t. A. Thus Theorem 2.1 is applicable and results in

$$\neg \exists x \exists z \Big[\\ R([\textsc{Gfp}\, A(t).x \neq t \vee \exists y \big(R(A(u), \neg A(v), x, y) \wedge A(y)\big)](w), \textsc{False}, x, z) \wedge \\ [\textsc{Gfp}\, A(t).x \neq t \vee \exists y \big(R(A(u), \neg A(v), x, y) \wedge A(y)\big)](z)\Big],$$

equivalent to

$$\forall x \forall z \Big[\\ R([\textsc{Gfp}\, A(t).x \neq t \vee \exists y \big(R(A(u), \neg A(v), x, y) \wedge A(y)\big)](w), \textsc{False}, x, z) \rightarrow \\ \neg[\textsc{Gfp}\, A(t).x \neq t \vee \exists y \big(R(A(u), \neg A(v), x, y) \wedge A(y)\big)](z)\Big],$$

being a condition on the class of frames validating the axiom (19). ◁

4. Conclusions

In the current paper we have extended second-order quantifier elimination techniques based on the Ackermann's lemma [1] and the fixpoint theorem of [17]. We then applied the introduced technique to a semantical analysis of conditionals depending on computing correspondences between axioms and Gabbay's third-order accessibility relation [10].

We expect that similar methodology can be applied in the case of other logics, since logical connectives and operators are actually of third-order and making them second- or first-order often requires nontrivial techniques and sometimes is impossible.

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