Representing Independence Models with Elementary Triplets

Jose M. Peña
ADIT, IDA, Linköping University, Sweden
jose.m.pena@liu.se

Abstract

An elementary triplet in an independence model represents a conditional independence statement between two singletons. It is known that these triplets can be used to represent the independence model unambiguously under some conditions. In this paper, we show how this representation helps performing efficiently some operations with independence models, such as finding the dominant triplets or a minimal independence map of an independence model, or computing the intersection or union of a pair of independence models.

1 Representation

Let V denote a finite set of elements. Subsets of V are denoted by upper-case letters, whereas the elements of V are denoted by lower-case letters. Given three sets $I, J, K \subseteq V$, the triplet $I \perp J|K$ denotes that I and J are conditionally independent given K. Given a set of triplets G, also known as an independence model, $I \perp_G J|K$ denotes that $I \perp J|K$ is in G. A triplet $I \perp J|K$ is called elementary if $|I| = |J| = 1$. We shall not distinguish between elements of V and singletons. We use IJ to denote $I \cup J$. Union has higher priority than set difference in expressions. Consider the following properties:

(CI0) $I \perp J|K \Leftrightarrow J \perp I|K$.
(CI1) $I \perp J|KL, I \perp K|L \Leftrightarrow I \perp JK|L$.
(CI2) $I \perp J|KL, I \perp K|JL \Rightarrow I \perp J|L, I \perp K|L$.
(CI3) $I \perp J|KL, I \perp K|JL \Leftarrow I \perp J|L, I \perp K|L$.

A set of triplets with the properties CI0-1/CI0-2/CI0-3 is also called a semigraphoid/graphoid/ compositional graphoid.[1] The CI0 property is also called symmetry property. The $\Rightarrow$ part of the CI1 property is also called contraction

[1] For instance, the independencies in a probability distribution form a semigraphoid, while the independencies in a strictly positive probability distribution form a graphoid, and the independencies in a regular Gaussian distribution form a compositional graphoid.

property, and the $\Leftarrow$ part corresponds to the so-called weak union and decomposition properties. The CI2 and CI3 properties are also called intersection and composition properties.[2] In addition, consider the following properties:

(ci0) $i \perp j|k \Leftrightarrow j \perp i|k$.
(ci1) $i \perp j|kL, i \perp k|L \Leftrightarrow i \perp k|jL, i \perp j|L$.
(ci2) $i \perp j|kL, i \perp k|jL \Rightarrow i \perp j|L, i \perp k|L$.
(ci3) $i \perp j|kL, i \perp k|jL \Leftarrow i \perp j|L, i \perp k|L$.

Note that CI2 and CI3 only differ in the direction of the implication. The same holds for ci2 and ci3.

Given a set of triplets $G = \{I \perp J|K\}$, let $\mathbb{P} = p(G) = \{i \perp j|M : I \perp_G J|K$ with $i \in I$, $j \in J$ and $K \subseteq M \subseteq (I \smallsetminus i)(J \smallsetminus j)K\}$. Given a set of elementary triplets $P = \{i \perp j|K\}$, let $\mathbb{G} = g(P) = \{I \perp J|K : i \perp_P j|M$ for all $i \in I$, $j \in J$ and $K \subseteq M \subseteq (I \smallsetminus i)(J \smallsetminus j)K\}$. The following two lemmas prove that there is a bijection between certain sets of triplets and certain sets of elementary triplets. The lemmas have been proven when G and P satisfy CI0-1 and ci0-1 [6, Proposition 1]. We extend them to the cases where G and P satisfy CI0-2/CI0-3 and ci0-2/ci0-3.

Lemma 1. *If G satisfies CI0-1/CI0-2/CI0-3 then (a) $\mathbb{P}$ satisfies ci0-1/ci0-2/ci0-3, (b) $G = g(\mathbb{P})$, and (c) $\mathbb{P} = \{i \perp j|K : i \perp_G j|K\}$.*

Proof. The proof of (c) is trivial. We now prove (a). That G satisfies C0 implies that $\mathbb{P}$ satisfies ci0 by definition of $\mathbb{P}$.

Proof of CI1 $\Rightarrow$ ci1

Since ci1 is symmetric, it suffices to prove the $\Rightarrow$ implication of ci1.

1. Assume that $i \perp_{\mathbb{P}} j|kL$.
2. Assume that $i \perp_{\mathbb{P}} k|L$.
3. Then, it follows from (1) and the definition of $\mathbb{P}$ that $i \perp_G j|kL$ or $I \perp_G J|M$ with $i \in I$, $j \in J$ and $M \subseteq kL \subseteq (I \smallsetminus i)(J \smallsetminus j)M$. Note that the latter case implies that $i \perp_G j|kL$ by CI1.
4. Then, $i \perp_G k|L$ by the same reasoning as in (3).
5. Then, $i \perp_G jk|L$ by CI1 on (3) and (4), which implies $i \perp_G k|jL$ and $i \perp_G j|L$ by CI1. Then, $i \perp_{\mathbb{P}} k|jL$ and $i \perp_{\mathbb{P}} j|L$ by definition of $\mathbb{P}$.

Proof of CI1-2 $\Rightarrow$ ci1-2

Assume that $i \perp_{\mathbb{P}} j|kL$ and $i \perp_{\mathbb{P}} k|jL$. Then, $i \perp_G j|kL$ and $i \perp_G k|jL$ by the same reasoning as in (3), which imply $i \perp_G j|L$ and $i \perp_G k|L$ by CI2. Then, $i \perp_{\mathbb{P}} j|L$ and $i \perp_{\mathbb{P}} k|L$ by definition of $\mathbb{P}$.

---

[2] Intersection is typically defined as $I \perp J|KL, I \perp K|JL \Rightarrow I \perp JK|L$. Note however that this and our definition are equivalent if CI1 holds. First, $I \perp JK|L$ implies $I \perp J|L$ and $I \perp K|L$ by CI1. Second, $I \perp J|L$ together with $I \perp K|JL$ imply $I \perp JK|L$ by CI1. Likewise, composition is typically defined as $I \perp JK|L \Leftarrow I \perp J|L, I \perp K|L$. Again, this and our definition are equivalent if CI1 holds. First, $I \perp JK|L$ implies $I \perp J|KL$ and $I \perp K|JL$ by CI1. Second, $I \perp K|JL$ together with $I \perp J|L$ imply $I \perp JK|L$ by CI1. In this paper, we will study sets of triplets that satisfy CI0-1, CI0-2 or CI0-3. So, the standard and our definitions are equivalent.

Proof of CI1-3 $\Rightarrow$ ci1-3

Assume that $i \perp_{\mathbb{P}} j|L$ and $i \perp_{\mathbb{P}} k|L$. Then, $i \perp_G j|L$ and $i \perp_G k|L$ by the same reasoning as in (3), which imply $i \perp_G j|kL$ and $i \perp_G k|jL$ by CI3. Then, $i \perp_{\mathbb{P}} j|kL$ and $i \perp_{\mathbb{P}} k|jL$ by definition of $\mathbb{P}$.

Finally, we prove (b). Clearly, $G \subseteq g(\mathbb{P})$ by definition of $\mathbb{P}$. To see that $g(\mathbb{P}) \subseteq G$, note that $I \perp_{g(\mathbb{P})} J|K \Rightarrow I \perp_G J|K$ holds when $|I| = |J| = 1$. Assume as induction hypothesis that the result also holds when $2 < |IJ| < s$. Assume without loss of generality that $1 < |J|$. Let $J = J_1 J_2$ st $J_1, J_2 \neq \varnothing$ and $J_1 \cap J_2 = \varnothing$. Then, $I \perp_{g(\mathbb{P})} J_1|K$ and $I \perp_{g(\mathbb{P})} J_2|J_1 K$ by ci1 and, thus, $I \perp_G J_1|K$ and $I \perp_G J_2|J_1 K$ by the induction hypothesis, which imply $I \perp_G J|K$ by CI1. $\square$

Lemma 2. *If P satisfies ci0-1/ci0-2/ci0-3 then (a) $\mathbb{G}$ satisfies CI0-1/CI0-2/CI0-3, (b) $P = p(\mathbb{G})$, and (c) $P = \{i \perp j|K : i \perp_{\mathbb{G}} j|K\}$.*

Proof. The proofs of (b) and (c) are trivial. We prove (a) below. That $\mathbb{P}$ satisfies ci0 implies that G satisfies C0 by definition of G.

Proof of ci1 $\Rightarrow$ CI1

The $\Leftarrow$ implication of CI1 is trivial. We prove below the $\Rightarrow$ implication.

1. Assume that $I \perp_{\mathbb{G}} j|KL$.
2. Assume that $I \perp_{\mathbb{G}} K|L$.
3. Let $i \in I$. Note that if $i \not\perp_P j|M$ with $L \subseteq M \subseteq (I \setminus i)KL$ then (c) $i \not\perp_P j|kM$ with $k \in K \setminus M$, and (d) $i \not\perp_P j|KM$. To see (c), assume to the contrary that $i \perp_P j|kM$. This together with $i \perp_P k|M$ (which follows from (2) by definition of $\mathbb{G}$) imply that $i \perp_P j|M$ by ci1, which contradicts the assumption of $i \not\perp_P j|M$. To see (d), note that $i \not\perp_P j|M$ implies $i \not\perp_P j|kM$ with $k \in K \setminus M$ by (c), which implies $i \not\perp_P j|kk'M$ with $k' \in K \setminus kM$ by (c) again, and so on until the desired result is obtained.
4. Then, $i \perp_P j|M$ for all $i \in I$ and $L \subseteq M \subseteq (I \setminus i)KL$. To see it, note that $i \perp_P j|KM$ follows from (1) by definition of $\mathbb{G}$, which implies the desired result by (d) in (3).
5. $i \perp_P k|M$ for all $i \in I$, $k \in K$ and $L \subseteq M \subseteq (I \setminus i)(K \setminus k)L$ follows from (2) by definition of $\mathbb{G}$.
6. $i \perp_P k|jM$ for all $i \in I$, $k \in K$ and $L \subseteq M \subseteq (I \setminus i)(K \setminus k)L$ follows from ci1 on (4) and (5).
7. $I \perp_{\mathbb{G}} jK|L$ follows from (4)-(6) by definition of $\mathbb{G}$.

Therefore, we have proven above the $\Rightarrow$ implication of CI1 when $|J| = 1$. Assume as induction hypothesis that the result also holds when $1 < |J| < s$. Let $J = J_1 J_2$ st $J_1, J_2 \neq \varnothing$ and $J_1 \cap J_2 = \varnothing$.

8. $I \perp_{\mathbb{G}} J_1|KL$ follows from $I \perp_{\mathbb{G}} J|KL$ by definition of $\mathbb{G}$.
9. $I \perp_{\mathbb{G}} J_2|J_1 KL$ follows from $I \perp_{\mathbb{G}} J|KL$ by definition of $\mathbb{G}$.
10. $I \perp_{\mathbb{G}} J_1 K|L$ by the induction hypothesis on (8) and $I \perp_{\mathbb{G}} K|L$.
11. $I \perp_{\mathbb{G}} JK|L$ by the induction hypothesis on (9) and (10).

Proof of ci1-2 $\Rightarrow$ CI1-2

12. Assume that $I \perp_{\mathbb{G}} j|kL$ and $I \perp_{\mathbb{G}} k|jL$.
13. $i \perp_P j|kM$ and $i \perp_P k|jM$ for all $i \in I$ and $L \subseteq M \subseteq (I \setminus i)L$ follows from (12) by definition of $\mathbb{G}$.
14. $i \perp_P j|M$ and $i \perp_P k|M$ for all $i \in I$ and $L \subseteq M \subseteq (I \setminus i)L$ by ci2 on (13).
15. $I \perp_{\mathbb{G}} j|L$ and $I \perp_{\mathbb{G}} k|L$ follows from (14) by definition of $\mathbb{G}$.

Therefore, we have proven the result when $|J| = |K| = 1$. Assume as induction hypothesis that the result also holds when $2 < |JK| < s$. Assume without loss of generality that $1 < |J|$. Let $J = J_1 J_2$ st $J_1, J_2 \neq \varnothing$ and $J_1 \cap J_2 = \varnothing$.

16. $I \perp_{\mathbb{G}} J_1|J_2KL$ and $I \perp_{\mathbb{G}} J_2|J_1KL$ by CI1 on $I \perp_{\mathbb{G}} J|KL$.
17. $I \perp_{\mathbb{G}} J_1|J_2L$ and $I \perp_{\mathbb{G}} J_2|J_1L$ by the induction hypothesis on (16) and $I \perp_{\mathbb{G}} K|JL$.
18. $I \perp_{\mathbb{G}} J|L$ by the induction hypothesis on (17).
19. $I \perp_{\mathbb{G}} K|L$ by CI1 on (18) and $I \perp_{\mathbb{G}} K|JL$.

Proof of ci1-3 $\Rightarrow$ CI1-3

20. Assume that $I \perp_{\mathbb{G}} j|L$ and $I \perp_{\mathbb{G}} k|L$.
21. $i \perp_P j|M$ and $i \perp_P k|M$ for all $i \in I$ and $L \subseteq M \subseteq (I \setminus i)L$ follows from (20) by definition of $\mathbb{G}$.
22. $i \perp_P j|kM$ and $i \perp_P k|jM$ for all $i \in I$ and $L \subseteq M \subseteq (I \setminus i)L$ by ci3 on (21).
23. $I \perp_{\mathbb{G}} j|kL$ and $I \perp_{\mathbb{G}} k|jL$ follows from (22) by definition of $\mathbb{G}$.

Therefore, we have proven the result when $|J| = |K| = 1$. Assume as induction hypothesis that the result also holds when $2 < |JK| < s$. Assume without loss of generality that $1 < |J|$. Let $J = J_1 J_2$ st $J_1, J_2 \neq \varnothing$ and $J_1 \cap J_2 = \varnothing$.

24. $I \perp_{\mathbb{G}} J_1|L$ by CI1 on $I \perp_{\mathbb{G}} J|L$.
25. $I \perp_{\mathbb{G}} J_2|J_1L$ by CI1 on $I \perp_{\mathbb{G}} J|L$.
26. $I \perp_{\mathbb{G}} K|J_1L$ by the induction hypothesis on (24) and $I \perp_{\mathbb{G}} K|L$.
27. $I \perp_{\mathbb{G}} K|JL$ by the induction hypothesis on (25) and (26).
28. $I \perp_{\mathbb{G}} JK|L$ by CI1 on (27) and $I \perp_{\mathbb{G}} J|L$.
29. $I \perp_{\mathbb{G}} J|KL$ and $I \perp_{\mathbb{G}} K|JL$ by CI1 on (28).

$\square$

The following two lemmas generalize Lemmas 1 and 2 by removing the assumptions about G and P.

Lemma 3. *Let G^* denote the CI0-1/CI0-2/CI0-3 closure of G, and let $\mathbb{P}^*$ denote the ci0-1/ci0-2/ci0-3 closure of $\mathbb{P}$. Then, $\mathbb{P}^* = p(G^*)$, $G^* = g(\mathbb{P}^*)$ and $\mathbb{P}^* = \{i \perp j|K : i \perp_{G^*} j|K\}$.*

Proof. Clearly, $G \subseteq g(\mathbb{P}^*)$ and, thus, $G^* \subseteq g(\mathbb{P}^*)$ because $g(\mathbb{P}^*)$ satisfies CI0-1/CI0-2/CI0-3 by Lemma 2. Clearly, $\mathbb{P} \subseteq p(G^*)$ and, thus, $\mathbb{P}^* \subseteq p(G^*)$ because $p(G^*)$ satisfies ci0-1/ci0-2/ci0-3 by Lemma 1. Then, $G^* \subseteq g(\mathbb{P}^*) \subseteq g(p(G^*))$ and $\mathbb{P}^* \subseteq p(G^*) \subseteq p(g(\mathbb{P}^*))$. Then, $G^* = g(\mathbb{P}^*)$ and $\mathbb{P}^* = p(G^*)$, because $G^* = g(p(G^*))$ and $\mathbb{P}^* = p(g(\mathbb{P}^*))$ by Lemmas 1 and 2. Finally, that $\mathbb{P}^* = \{i \perp j|K : i \perp_{G^*} j|K\}$ is now trivial. □

Lemma 4. *Let P^* denote the ci0-1/ci0-2/ci0-3 closure of P, and let $\mathbb{G}^*$ denote the CI0-1/CI0-2/CI0-3 closure of $\mathbb{G}$. Then, $\mathbb{G}^* = g(P^*)$, $P^* = p(\mathbb{G}^*)$ and $P^* = \{i \perp j|K : i \perp_{\mathbb{G}^*} j|K\}$.*

Proof. Clearly, $P \subseteq p(\mathbb{G}^*)$ and, thus, $P^* \subseteq p(\mathbb{G}^*)$ because $p(\mathbb{G}^*)$ satisfies ci0-1/ci0-2/ci0-3 by Lemma 1. Clearly, $\mathbb{G} \subseteq g(P^*)$ and, thus, $\mathbb{G}^* \subseteq g(P^*)$ because $g(P^*)$ satisfies CI0-1/CI0-2/CI0-3 by Lemma 2. Then, $P^* \subseteq p(\mathbb{G}^*) \subseteq p(g(P^*))$ and $\mathbb{G}^* \subseteq g(P^*) \subseteq g(p(\mathbb{G}^*))$. Then, $P^* = p(\mathbb{G}^*)$ and $\mathbb{G}^* = g(P^*)$, because $P^* = p(g(P^*))$ and $\mathbb{G}^* = g(p(\mathbb{G}^*))$ by Lemmas 1 and 2. Finally, that $P^* = \{i \perp j|K : i \perp_{\mathbb{G}^*} j|K\}$ is now trivial. □

The parts (a) of Lemmas 1 and 2 imply that every set of triplets G satisfying CI0-1/CI0-2/CI0-3 can be paired to a set of elementary triplets P satisfying ci0-1/ci0-2/ci0-3, and vice versa. The pairing is actually a bijection, due to the parts (b) of the lemmas. Thanks to this bijection, we can use $\mathbb{P}$ to represent G. This is in general a much more economical representation: If $|V| = n$, then there up to 4^n triplets,[3] whereas there are $n^2 \cdot 2^{n-2}$ elementary triplets at most. We can reduce further the size of the representation by iteratively removing from $\mathbb{P}$ an elementary triplet that follows from two others by ci0-1/ci0-2/ci0-3. Note that $\mathbb{P}$ is an unique representation of G but the result of the removal process is not in general, as ties may occur during the process.

Likewise, Lemmas 3 and 4 imply that there is a bijection between the CI0-1/CI0-2/CI0-3 closures of sets of triplets and the ci0-1/ci0-2/ci0-3 closures of sets of elementary triplets. Thanks to this bijection, we can use $\mathbb{P}^*$ to represent G^*. Note that $\mathbb{P}^*$ is obtained by ci0-1/ci0-2/ci0-3 closing $\mathbb{P}$, which is obtained from G. So, there is no need to CI0-1/CI0-2/CI0-3 close G and so produce G^*. Whether closing $\mathbb{P}$ can be done faster than closing G on average is an open question. In the worst-case scenario, both imply applying the corresponding properties a number of times exponential in $|V|$ [7]. We can avoid this problem by simply using $\mathbb{P}$ to represent G^*, because $\mathbb{P}$ is the result of running the removal process outline above on $\mathbb{P}^*$. All the results in the sequel assume that G and P satisfy CI0-1/CI0-2/CI0-3 and ci0-1/ci0-2/ci0-3. Thanks to Lemmas 3 and 4, these assumptions can be dropped by replacing G, P, $\mathbb{G}$ and $\mathbb{P}$ in the results below with G^*, P^*, $\mathbb{G}^*$ and $\mathbb{P}^*$.

Let $I = i_1 \dots i_m$ and $J = j_1 \dots j_n$. In order to decide whether $I \perp_{\mathbb{G}} J|K$, the definition of $\mathbb{G}$ implies checking whether $m \cdot n \cdot 2^{(m+n-2)}$ elementary triplets are in P. The following lemma simplifies this for when P satisfies ci0-1, as it implies

[3] A triplet can be represented as a n-tuple whose entries state if the corresponding node is in the first, second, third or none set of the triplet.

checking $m{\cdot}n$ elementary triplets. For when P satisfies ci0-2 or ci0-3, the lemma simplifies the decision even further as the conditioning sets of the elementary triplets checked have all the same size or form.

Lemma 5. *Let* $\mathbb{H}_1 = \{I \perp J|K : i_s \perp_P j_t|i_1 \ldots i_{s-1} j_1 \ldots j_{t-1} K$ *for all* $1 \leq s \leq m$ *and* $1 \leq t \leq n\}$, $\mathbb{H}_2 = \{I \perp J|K : i \perp_P j|(I \setminus i)(J \setminus j)K$ *for all* $i \in I$ *and* $j \in J\}$, *and* $\mathbb{H}_3 = \{I \perp J|K : i \perp_P j|K$ *for all* $i \in I$ *and* $j \in J\}$. *If* P *satisfies ci0-1, then* $\mathbb{G} = \mathbb{H}_1$. *If* P *satisfies ci0-2, then* $\mathbb{G} = \mathbb{H}_2$. *If* P *satisfies ci0-3, then* $\mathbb{G} = \mathbb{H}_3$.

Proof. Proof for ci0-1

It suffices to prove that $\mathbb{H}_1 \subseteq \mathbb{G}$, because it is clear that $\mathbb{G} \subseteq \mathbb{H}_1$. Assume that $I \perp_{\mathbb{H}_1} J|K$. Then, $i_s \perp_P j_t|i_1 \ldots i_{s-1} j_1 \ldots j_{t-1} K$ and $i_s \perp_P j_{t+1}|i_1 \ldots i_{s-1} j_1 \ldots j_t K$ by definition of $\mathbb{H}_1$. Then, $i_s \perp_P j_{t+1}|i_1 \ldots i_{s-1} j_1 \ldots j_{t-1} K$ and $i_s \perp_P j_t|i_1 \ldots i_{s-1} j_1 \ldots j_{t-1} j_{t+1} K$ by ci1. Then, $i_s \perp_{\mathbb{G}} j_{t+1}|i_1 \ldots i_{s-1} j_1 \ldots j_{t-1} K$ and $i_s \perp_{\mathbb{G}} j_t|i_1 \ldots i_{s-1} j_1 \ldots j_{t-1} j_{t+1} K$ by definition of $\mathbb{G}$. By repeating this reasoning, we can then conclude that $i_s \perp_{\mathbb{G}} j_{\sigma(t)}|i_1 \ldots i_{s-1} j_{\sigma(1)} \ldots j_{\sigma(t-1)} K$ for any permutation σ of the set $\{1 \ldots n\}$. By following an analogous reasoning for i_s instead of j_t, we can then conclude that $i_{\varsigma(s)} \perp_{\mathbb{G}} j_{\sigma(t)}|i_{\varsigma(1)} \ldots i_{\varsigma(s-1)} j_{\sigma(1)} \ldots j_{\sigma(t-1)} K$ for any permutations σ and ς of the sets $\{1 \ldots n\}$ and $\{1 \ldots m\}$. This implies the desired result by definition of $\mathbb{G}$.

Proof for ci0-2

It suffices to prove that $\mathbb{H}_2 \subseteq \mathbb{G}$, because it is clear that $\mathbb{G} \subseteq \mathbb{H}_2$. Note that $\mathbb{G}$ satisfies CI0-2 by Lemma 2. Assume that $I \perp_{\mathbb{H}_2} J|K$.

1. $i_1 \perp_{\mathbb{G}} j_1|(I \setminus i_1)(J \setminus j_1)K$ and $i_1 \perp_{\mathbb{G}} j_2|(I \setminus i_1)(J \setminus j_2)K$ follow from $i_1 \perp_P j_1|(I \setminus i_1)(J \setminus j_1)K$ and $i_1 \perp_P j_2|(I \setminus i_1)(J \setminus j_2)K$ by definition of $\mathbb{G}$.
2. $i_1 \perp_{\mathbb{G}} j_1|(I \setminus i_1)(J \setminus j_1 j_2)K$ by CI2 on (1), which together with (1) imply $i_1 \perp_{\mathbb{G}} j_1 j_2|(I \setminus i_1)(J \setminus j_1 j_2)K$ by CI1.
3. $i_1 \perp_{\mathbb{G}} j_3|(I \setminus i_1)(J \setminus j_3)K$ follows from $i_1 \perp_P j_3|(I \setminus i_1)(J \setminus j_3)K$ by definition of $\mathbb{G}$.
4. $i_1 \perp_{\mathbb{G}} j_1 j_2|(I \setminus i_1)(J \setminus j_1 j_2 j_3)K$ by CI2 on (2) and (3), which together with (3) imply $i_1 \perp_{\mathbb{G}} j_1 j_2 j_3|(I \setminus i_1)(J \setminus j_1 j_2 j_3)K$ by CI1.

By continuing with the reasoning above, we can conclude that $i_1 \perp_{\mathbb{G}} J|(I \setminus i_1)K$. By an analogous reasoning, we can conclude that $i_1 i_2 \perp_{\mathbb{G}} J|(I \setminus i_1 i_2)K$, $i_1 i_2 i_3 \perp_{\mathbb{G}} J|(I \setminus i_1 i_2 i_3)K$ and so on until the desired is obtained.

Proof for ci0-3

It suffices to prove that $\mathbb{H}_3 \subseteq \mathbb{G}$, because it is clear that $\mathbb{G} \subseteq \mathbb{H}_3$. Note that $\mathbb{G}$ satisfies CI0-3 by Lemma 2. Assume that $I \perp_{\mathbb{H}_3} J|K$.

1. $i_1 \perp_{\mathbb{G}} j_1|K$ and $i_1 \perp_{\mathbb{G}} j_2|K$ follow from $i_1 \perp_P j_1|K$ and $i_1 \perp_P j_2|K$ by definition of $\mathbb{G}$.
2. $i_1 \perp_{\mathbb{G}} j_1|j_2 K$ by CI3 on (1), which together with (1) imply $i_1 \perp_{\mathbb{G}} j_1 j_2|K$ by CI1.
3. $i_1 \perp_{\mathbb{G}} j_3|K$ follows from $i_1 \perp_P j_3|K$ by definition of $\mathbb{G}$.
4. $i_1 \perp_{\mathbb{G}} j_1 j_2|j_3 K$ by CI3 on (2) and (3), which together with (3) imply $i_1 \perp_{\mathbb{G}} j_1 j_2 j_3|K$ by CI1.

By continuing with the reasoning above, we can conclude that $i_1 \perp_{\mathbb{G}} J|K$. By an analogous reasoning, we can conclude that $i_1 i_2 \perp_{\mathbb{G}} J|K$, $i_1 i_2 i_3 \perp_{\mathbb{G}} J|K$ and so on until the desired result is obtained. $\square$

We are not the first to use some distinguished triplets of G to represent it. However, most other works use dominant triplets for this purpose [1, 4, 5, 9]. The following lemma shows how to find these triplets with the help of $\mathbb{P}$. A triplet $I \perp J|K$ dominates another triplet $I' \perp J'|K'$ if $I' \subseteq I$, $J' \subseteq J$ and $K \subseteq K' \subseteq (I \setminus I')(J \setminus J')K$. Given a set of triplets, a triplet in the set is called dominant if no other triplet in the set dominates it.

Lemma 6. *If G satisfies CI0-1, then $I \perp J|K$ is a dominant triplet in G iff $I = i_1 \dots i_m$ and $J = j_1 \dots j_n$ are two maximal sets st $i_s \perp_{\mathbb{P}} j_t | i_1 \dots i_{s-1} j_1 \dots j_{t-1} K$ for all $1 \leq s \leq m$ and $1 \leq t \leq n$ and, for all $k \in K$, $i_s \not\perp_{\mathbb{P}} k | i_1 \dots i_{s-1} J(K \setminus k)$ and $k \not\perp_{\mathbb{P}} j_t | I j_1 \dots j_{t-1}(K \setminus k)$ for some $1 \leq s \leq m$ and $1 \leq t \leq n$. If G satisfies CI0-2, then $I \perp J|K$ is a dominant triplet in G iff I and J are two maximal sets st $i \perp_{\mathbb{P}} j|(I \setminus i)(J \setminus j)K$ for all $i \in I$ and $j \in J$ and, for all $k \in K$, $i \not\perp_{\mathbb{P}} k|(I \setminus i)J(K \setminus k)$ and $k \not\perp_{\mathbb{P}} j|I(J \setminus j)(K \setminus k)$ for some $i \in I$ and $j \in J$. If G satisfies CI0-3, then $I \perp J|K$ is a dominant triplet in G iff I and J are two maximal sets st $i \perp_{\mathbb{P}} j|K$ for all $i \in I$ and $j \in J$ and, for all $k \in K$, $i \not\perp_{\mathbb{P}} k|K \setminus k$ and $k \not\perp_{\mathbb{P}} j|K \setminus k$ for some $i \in I$ and $j \in J$.*

Proof. We proof the lemma for when G satisfies CI0-1. The other two cases can be proven in much the same way. To see the if part, note that $I \perp_G J|K$ by Lemmas 1 and 5. Moreover, assume to the contrary that there is a triplet $I' \perp_G J'|K'$ that dominates $I \perp_G J|K$. Consider the following two cases: $K' = K$ and $K' \subset K$. In the first case, CI1 on $I' \perp_G J'|K'$ implies that $I i_{m+1} \perp_G J|K$ or $I \perp_G J j_{n+1}|K$ with $i_{m+1} \in I' \setminus I$ and $j_{n+1} \in J' \setminus J$. Assume the latter without loss of generality. Then, CI1 implies that $i_s \perp_{\mathbb{P}} j_t | i_1 \dots i_{s-1} j_1 \dots j_{t-1} K$ for all $1 \leq s \leq m$ and $1 \leq t \leq n+1$. This contradicts the maximality of J. In the second case, CI1 on $I' \perp_G J'|K'$ implies that $Ik \perp_G J|K \setminus k$ or $I \perp_G Jk|K \setminus k$ with $k \in K$. Assume the latter without loss of generality. Then, CI1 implies that $i_s \perp_{\mathbb{P}} k | i_1 \dots i_{s-1} J(K \setminus k)$ for all $1 \leq s \leq m$, which contradicts the assumptions of the lemma.

To see the only if part, note that CI1 implies that $i_s \perp_{\mathbb{P}} j_t | i_1 \dots i_{s-1} j_1 \dots j_{t-1} K$ for all $1 \leq s \leq m$ and $1 \leq t \leq n$. Moreover, assume to the contrary that for some $k \in K$, $i_s \perp_{\mathbb{P}} k | i_1 \dots i_{s-1} J(K \setminus k)$ for all $1 \leq s \leq m$ or $k \perp_{\mathbb{P}} j_t | I j_1 \dots j_{t-1}(K \setminus k)$ for all $1 \leq t \leq n$. Assume the latter without loss of generality. Then, $Ik \perp_G J|K \setminus k$ by Lemmas 1 and 5, which implies that $I \perp_G J|K$ is not a dominant triplet in G, which is a contradiction. Finally, note that I and J must be maximal sets satisfying the properties proven in this paragraph because, otherwise, the previous paragraph implies that there is a triplet in G that dominates $I \perp_G J|K$. $\square$

Inspired by [7], if G satisfies CI0-1 then we represent $\mathbb{P}$ as a DAG. The nodes of the DAG are the elementary triplets in $\mathbb{P}$ and the edges of the DAG are $\{i \perp_{\mathbb{P}} k|L \rightarrow i \perp_{\mathbb{P}} j|kL\} \cup \{k \perp_{\mathbb{P}} j|L \dashrightarrow i \perp_{\mathbb{P}} j|kL\}$. See Figure 1 for an example.

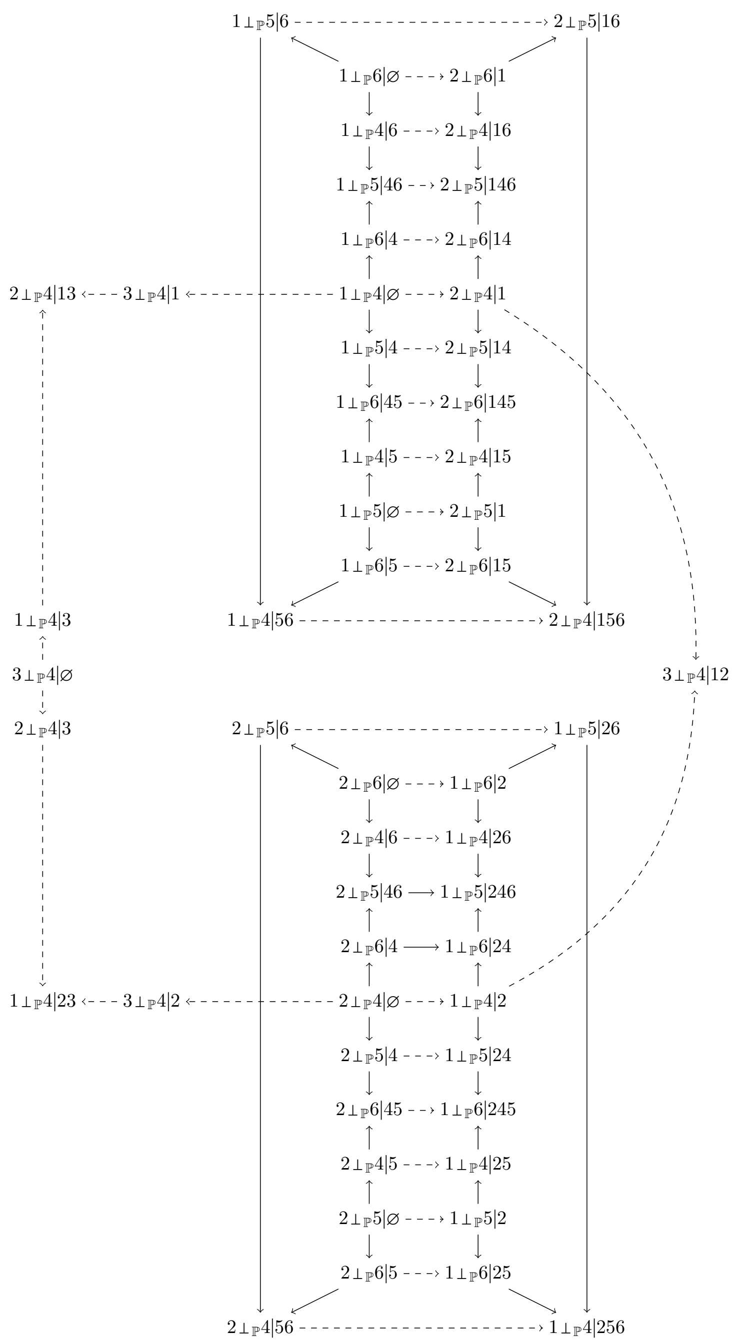

Figure 1: DAG representation of $\mathbb{P}$ (up to symmetry).

For the sake of readability, the DAG in the figure does not include symmetric elementary triplets. That is, the complete DAG can be obtained by adding a second copy of the DAG in the figure, replacing every node $i \perp_{\mathbb{P}} j|K$ in the copy with $j \perp_{\mathbb{P}} i|K$, and replacing every edge $\rightarrow$ in the copy with $\dashrightarrow$. We say that a subgraph over $m \cdot n$ nodes of the DAG is a grid if there is a bijection between the nodes of the subgraph and the labels $\{v_{s,t} : 1 \leq s \leq m, 1 \leq t \leq n\}$ st the edges of the subgraph are $\{v_{s,t} \rightarrow v_{s,t+1} : 1 \leq s \leq m, 1 \leq t < n\} \cup \{v_{s,t} \dashrightarrow v_{s+1,t} : 1 \leq s < m, 1 \leq t \leq n\}$. For instance, the following subgraph of the DAG in Figure 1 is a grid.

$$\begin{tikzcd}
2 \perp_{\mathbb{P}} 5|4 \arrow[r, dashed] \arrow[d] & 1 \perp_{\mathbb{P}} 5|24 \arrow[d] \\
2 \perp_{\mathbb{P}} 6|45 \arrow[r, dashed] & 1 \perp_{\mathbb{P}} 6|245
\end{tikzcd}$$

The following lemma is an immediate consequence of Lemmas 1 and 5.

Lemma 7. *Let G satisfy CI0-1, and let $I = i_1 \ldots i_m$ and $J = j_1 \ldots j_n$. If the subgraph of the DAG representation of $\mathbb{P}$ induced by the set of nodes $\{i_s \perp_{\mathbb{P}} j_t | i_1 \ldots i_{s-1} j_1 \ldots j_{t-1} K : 1 \leq s \leq m, 1 \leq t \leq n\}$ is a grid, then $I \perp_G J|K$.*

Thanks to Lemmas 6 and 7, finding dominant triplets can now be reformulated as finding maximal grids in the DAG. Note that this is a purely graphical characterization. For instance, the DAG in Figure 1 has 18 maximal grids: The subgraphs induced by the set of nodes $\{\sigma(s) \perp_{\mathbb{P}} \varsigma(t) | \sigma(1) \ldots \sigma(s-1) \varsigma(1) \ldots \varsigma(t-1) : 1 \leq s \leq 2, 1 \leq t \leq 3\}$ where σ and ς are permutations of $\{1,2\}$ and $\{4,5,6\}$, and the set of nodes $\{\pi(s) \perp_{\mathbb{P}} 4 | \pi(1) \ldots \pi(s-1) : 1 \leq s \leq 3\}$ where π is a permutation of $\{1,2,3\}$. These grids correspond to the dominant triplets $12 \perp_G 456|\varnothing$ and $123 \perp_G 4|\varnothing$.

2 Operations

In this section, we discuss some operations with independence models that can efficiently be performed with the help of $\mathbb{P}$. See [2, 3] for how to perform these operations efficiently when independence models are represented by their dominant triplets.

2.1 Membership

We want to check whether $I \perp_G J|K$, where G denotes a set of triplets satisfying CI0-1/CI0-2/CI0-3. Recall that G can be obtained from $\mathbb{P}$ by Lemma 1. Recall also that $\mathbb{P}$ satisfies ci0-1/ci0-2/ci0-3 by Lemma 1 and, thus, Lemma 5 applies to $\mathbb{P}$, which simplifies producing G from $\mathbb{P}$. Specifically if G satisfies CI0-1, then we can check whether $I \perp_G J|K$ with $I = i_1 \ldots i_m$ and $J = j_1 \ldots j_n$ by checking whether $i_s \perp_{\mathbb{P}} j_t | i_1 \ldots i_{s-1} j_1 \ldots j_{t-1} K$ for all $1 \leq s \leq m$ and $1 \leq t \leq n$. Thanks to Lemma 7, this solution can also be reformulated as checking whether the DAG representation of $\mathbb{P}$ contains a suitable grid. Likewise, if G satisfies CI0-2, then we can check whether $I \perp_G J|K$ by checking whether $i \perp_{\mathbb{P}} j | (I \setminus i)(J \setminus j) K$ for all $i \in I$ and $j \in J$. Finally, if G satisfies CI0-3, then we can check whether $I \perp_G J|K$ by checking whether $i \perp_{\mathbb{P}} j|K$ for all $i \in I$ and $j \in J$. Note that in the last two cases, we only need to check elementary triplets with conditioning sets of a specific length or form.

2.2 Minimal Independence Map

We say that a DAG D is a minimal independence map (MIM) of a set of triplets G relative to an ordering σ of the elements in V if (i) $I \perp_D J|K \Rightarrow I \perp_G J|K$,[4] (ii) removing any edge from D makes it cease to satisfy condition (i), and (iii) the edges of D are of the form $\sigma(s) \to \sigma(t)$ with $s < t$. If G satisfies CI0-1, then D can be built by setting $Pa_D(\sigma(s))$[5] for all $1 \le s \le |V|$ to a minimal subset of $\sigma(1)\dots\sigma(s-1)$ st $\sigma(s) \perp_G \sigma(1)\dots\sigma(s-1) \setminus Pa_D(\sigma(s))|Pa_D(\sigma(s))$ [8, Theorem 9]. Thanks to Lemma 7, building a MIM of G relative to σ can now be reformulated as finding, for all $1 \le s \le |V|$, a longest grid in the DAG representation of $\mathbb{P}$ that is of the form $\sigma(s) \perp_{\mathbb{P}} j_1|\sigma(1)\dots\sigma(s-1) \setminus j_1 \dots j_n \to \sigma(s) \perp_{\mathbb{P}} j_2|\sigma(1)\dots\sigma(s-1) \setminus j_2 \dots j_n \to \dots \to \sigma(s) \perp_{\mathbb{P}} j_n|\sigma(1)\dots\sigma(s-1) \setminus j_n$, or $j_1 \perp_{\mathbb{P}} \sigma(s)|\sigma(1)\dots\sigma(s-1) \setminus j_1 \dots j_n \dashrightarrow j_2 \perp_{\mathbb{P}} \sigma(s)|\sigma(1)\dots\sigma(s-1) \setminus j_2 \dots j_n \dashrightarrow \dots \dashrightarrow j_n \perp_{\mathbb{P}} \sigma(s)|\sigma(1)\dots\sigma(s-1) \setminus j_n$ with $j_1 \dots j_n \subseteq \sigma(1)\dots\sigma(s-1)$. Then, we set $Pa_D(\sigma(s))$ to $\sigma(1)\dots\sigma(s-1) \setminus j_1 \dots j_n$.

We say that a DAG D is a perfect map (PM) of a set of triplets G if $I \perp_D J|K \Leftrightarrow I \perp_G J|K$. We can check whether G has a PM with the help of $\mathbb{P}$ as follows: G has a PM iff $PM(\emptyset, \emptyset)$ returns true, where

$\underline{PM(Visited, Marked)}$

if $Visited = V$ then
  if all the nodes in the DAG representation of $\mathbb{P}$ are in $Marked$ then return true and stop
else
  for each node $i \in V \setminus Visited$ do
    for each longest grid in the DAG representation of $\mathbb{P}$ that is of the form
    $i \perp_{\mathbb{P}} j_1|Visited \setminus j_1 \dots j_n \to i \perp_{\mathbb{P}} j_2|Visited \setminus j_2 \dots j_n \to \dots \to i \perp_{\mathbb{P}} j_n|Visited \setminus j_n$ or
    $j_1 \perp_{\mathbb{P}} i|Visited \setminus j_1 \dots j_n \dashrightarrow j_2 \perp_{\mathbb{P}} i|Visited \setminus j_2 \dots j_n \dashrightarrow \dots \dashrightarrow j_n \perp_{\mathbb{P}} i|Visited \setminus j_n$ with
    $j_1 \dots j_n \subseteq Visited$ do
      $PM(Visited \cup \{i\},$
      $Marked \cup p(\{i \perp_G j_1 \dots j_n|Visited \setminus j_1 \dots j_n\}) \cup p(\{j_1 \dots j_n \perp_G i|Visited \setminus j_1 \dots j_n\}))$

2.3 Inclusion

Let G and G' denote two sets of triplets satisfying CI0-1/CI0-2/CI0-3. We can check whether $G \subseteq G'$ by checking whether $\mathbb{P} \subseteq \mathbb{P}'$. If the DAG representations of $\mathbb{P}$ and $\mathbb{P}'$ are available, then we can answer the inclusion question by checking whether the former is a subgraph of the latter.

[4] $I \perp_D J|K$ stands for I and J are d-separated in D given K.
[5] $Pa_D(\sigma(s))$ denotes the parents of $\sigma(s)$ in D.

2.4 Intersection

Let G and G' denote two sets of triplets satisfying CI0-1/CI0-2/CI0-3. Note that $G \cap G'$ satisfies CI0-1/CI0-2/CI0-3. Likewise, $\mathbb{P} \cap \mathbb{P}'$ satisfies ci0-1/ci0-2/ci0-3. We can represent $G \cap G$ by $\mathbb{P} \cap \mathbb{P}'$. To see it, note that $I \perp_{G \cap G'} J|K$ iff $i \perp_{\mathbb{P}} j|M$ and $i \perp_{\mathbb{P}'} j|M$ for all $i \in I$, $j \in J$, and $K \subseteq M \subseteq (I \setminus i)(J \setminus j)K$. If the DAG representations of $\mathbb{P}$ and $\mathbb{P}'$ are available, then we can represent $G \cap G$ by the subgraph of either of them induced by the nodes that are in both of them.

2.5 Union

Let G and G' denote two sets of triplets satisfying CI0-1/CI0-2/CI0-3. Note that $G \cup G'$ may not satisfy CI0-1/CI0-2/CI0-3. For instance, let $G = \{x \perp y|z, y \perp x|z\}$ and $G' = \{x \perp z|\varnothing, z \perp x|\varnothing\}$. We can solve this problem by simply adding an auxiliary element e (respectively e') to the conditioning set of every triplet in G (respectively G'). In the previous example, $G = \{x \perp y|ze, y \perp x|ze\}$ and $G' = \{x \perp z|e', z \perp x|e'\}$. Now, we can represent $G \cup G'$ by first adding the auxiliary element e (respectively e') to the conditioning set of every elementary triplet in $\mathbb{P}$ (respectively $\mathbb{P}'$) and, then, taking $\mathbb{P} \cup \mathbb{P}'$. This solution has advantages and disadvantages. The main advantage is that we represent $G \cup G'$ exactly. One of the disadvantages is that the same elementary triplet may appear twice in the representation, i.e. with e and e' in the conditioning set. Another disadvantage is that we need to modify slightly the procedures described above for building MIMs, and checking membership and inclusion. We believe that the advantage outweighs the disadvantages.

If the solution above is not satisfactory, then we have two options: Representing a minimal superset or a maximal superset of $G \cup G'$ satisfying CI0-1/CI0-2/CI0-3. Note that the minimal superset of $G \cup G'$ satisfying CI0-1/CI0-2/CI0-3 is unique because, otherwise, the intersection of any two such supersets is a superset of $G \cup G'$ that satisfies CI0-1/CI0-2/CI0-3, which contradicts the minimality of the original supersets. On the other hand, the maximal subset of $G \cup G'$ satisfying CI0-1/CI0-2/CI0-3 is not unique. For instance, let $G = \{x \perp y|z, y \perp x|z\}$ and $G' = \{x \perp z|\varnothing, z \perp x|\varnothing\}$. We can represent the minimal superset of $G \cup G'$ satisfying CI0-1/CI0-2/CI0-3 by the ci0-1/ci0-2/ci0-3 closure of $\mathbb{P} \cup \mathbb{P}'$. Clearly, this representation represents a superset of $G \cup G'$. Moreover, the superset satisfies CI0-1/CI0-2/CI0-3 by Lemma 2. Minimality follows from the fact that removing any elementary triplet from the representation implies not representing some triplet in $G \cup G'$ by Lemma 1. Note that the DAG representation of $G \cup G'$ is not the union of the DAG representations of $\mathbb{P}$ and $\mathbb{P}'$, because we first have to close $\mathbb{P} \cup \mathbb{P}'$ under ci0-1/ci0-2/ci0-3. We can represent a maximal subset of $G \cup G'$ satisfying CI0-1/CI0-2/CI0-3 by a maximal subset U of $\mathbb{P} \cup \mathbb{P}'$ that is closed under ci0-1/ci0-2/ci0-3 and st every triplet represented by U is in $G \cup G'$. Recall that we can efficiently check the latter as shown above. In fact, we do not need to check it for every triplet but only for the dominant triplets. Recall that these can efficiently be obtained from U as shown in the previous section.

3 Discussion

In this work, we have proposed to represent semigraphoids, graphoids and compositional graphoids by their elementary triplets. We have also shown how this representation helps performing efficiently some common operations between independence models. Whether this implies a gain of efficiency compared to other representations (e.g. dominant triplets) is a question for future research.

References

[1] Marco Baioletti, Giuseppe Busanello, and Barbara Vantaggi. Conditional independence structure and its closure: Inferential rules and algorithms. *International Journal of Approximate Reasoning*, 50(7):1097–1114, 2009.

[2] Marco Baioletti, Giuseppe Busanello, and Barbara Vantaggi. Acyclic directed graphs representing independence models. *International Journal of Approximate Reasoning*, 52(1):2 – 18, 2011.

[3] Marco Baioletti, Davide Petturiti, and Barbara Vantaggi. Qualitative combination of independence models. In *Proceedings of the 12th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, pages 37–48, 2013.

[4] Peter de Waal and Linda C. van der Gaag. Stable independence and complexity of representation. In *Proceedings of the 20th Conference on Uncertainty in Artificial Intelligence*, pages 112–119, 2004.

[5] Stavros Lopatatzidis and Linda C. van der Gaag. Computing concise representations of semi-graphoid independency models. In *Proceedings of the 13th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, pages 290–300, 2015.

[6] Frantisek Matús. Ascending and descending conditional independence relations. In *Proceedings of the 11th Prague Conference on Information Theory, Statistical Decision Functions and Random Processes*, pages 189–200, 1992.

[7] Frantisek Matús. Lengths of semigraphoid inferences. *Annals of Mathematics and Artificial Intelligence*, 35:287–294, 2002.

[8] Judea Pearl. *Probabilistic reasoning in intelligent systems: Networks of plausible inference*. Morgan Kaufmann Publishers Inc., 1988.

[9] Milan Studený. Complexity of structural models. In *Proceedings of the Joint Session of the 6th Prague Conference on Asymptotic Statistics and the 13th Prague Conference on Information Theory, Statistical Decision Functions and Random Processes*, pages 521–528, 1998.