

# Trichotomies in the Complexity of Minimal Inference

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## Abstract

We study the complexity of the propositional minimal inference problem. Although the complexity of this problem has been already extensively studied before because of its fundamental importance in nonmonotonic logics and commonsense reasoning, no complete classification of its complexity was found. We classify the complexity of four different and well-studied formalizations of the problem, proving that the complexity of the minimal inference problem for each of them has a trichotomy (between P, coNP-complete, and  $\Pi_2$ P-complete). One of these results finally settles with a positive answer the trichotomy conjecture of Kirousis and Kolaitis [A dichotomy in the complexity of propositional circumscription, LICS'01].

## 1 Introduction and Summary of Results

Reasoning with minimal models of a theory is a general idea widely used in artificial intelligence, especially for capturing various aspects of common sense and nonmonotonic reasoning. In particular, it is the main idea behind circumscription [19, 20], diagnosis [10], default logic [26], and logic programming under stable model semantics [13]. The key idea behind minimal models reasoning, and in particular circumscription, is that the minimal models have as few “exceptions” as possible, and embody therefore common sense. Minimal inference has been shown by Gelfond *et al.* [14] to coincide with reasoning under the extended closed world assumption, which is one of the main formalisms for reasoning with incomplete information. We focus in this paper on the important basic case where the theory is propositional, which is also the most investigated case from a complexity point of view. In this case, the minimality of models is defined with respect to the *pointwise partial order*, extending the order  $0 < 1$  on truth values.

The complexity of several basic algorithmic problems have been studied in connection with minimal models of propositional formulas: among them are the *model selection* [7, 22], *model checking* [5, 17], and *inference* [6, 11, 12, 18] problems. Given a propositional formula  $\varphi$ , the minimal model selection problem requires to compute a minimal model of  $\varphi$ . Similarly, given a propositional formula  $\varphi$  and a truth assignment  $m$ , the minimal model checking problem asks whether  $m$  is a minimal model of  $\varphi$ . Given two propositional formulas  $\varphi$  and  $\psi$ , the minimal inference problem asks whether  $\psi$  is true in every minimal model of  $\varphi$ . In propositional theories these problems are identical to the corresponding problems for propositional circumscription. Hence, they are often called

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the model selection, model checking, and inference problems for propositional circumscription, respectively. In this paper we focus exclusively on the minimal inference problem (or equivalently the inference problem for propositional circumscription).

McCarthy [20] and Gelfond, Przymusinska, and Przymusinski [14] extended the notion of circumscription to a more general setting. They altered the notion of minimal models of a formula  $\varphi$  by dividing the set of variables  $V$  of  $\varphi$  into three disjoint sets, namely of the variables  $P$  that are minimized, the variables  $Z$  that may vary, and the variables  $Q$  that maintain a fixed value. Given two models  $m$  and  $m'$  of  $\varphi$ , then  $m \leq m'$  if and only if  $m(x) \leq m'(x)$  for all  $x \in P$  and  $m(y) = m'(y)$  for all  $y \in Q$ . A model  $m'$  of  $\varphi$  is defined to be a minimal model if and only if there is no model  $m$  of  $\varphi$  such that  $m \leq m'$  and  $m(x) < m'(x)$  for a variable  $x \in P$ .

The generalized minimal inference problem with the extended notion of minimal models is called  $(P, Z, Q)$ -minimal inference or  $(P, Z, Q)$ -circumscription. It should be clear that the original circumscription and minimal inference problem is the special cases of the corresponding generalized problem satisfying the condition  $Z = Q = \emptyset$ . Cadoli and Lenzerini analyzed in [6] the complexity of  $(P, Z, Q)$ -minimal inference, as well as the variants with no fixed variables (i.e.,  $(P, Z, \emptyset)$ -minimal inference), with no free variables (i.e.,  $(P, \emptyset, Q)$ -minimal inference), and where all variables must be minimized (i.e.,  $(P, \emptyset, \emptyset)$ -minimal inference). In fact, they analyzed the complexity of reasoning under several variants of closed world assumptions and their results propagate to minimal inference through the equivalence presented by Gelfond *et al.* in [14].

The topic of this paper is to classify the complexity of the minimal inference problem for the four restrictions on the variable sets  $Q$  and  $Z$  mentioned above, as well as all restrictions on the types of clauses allowed in the theory  $\varphi$ , presented by a formula in conjunctive normal form. Note that the corresponding classification problem for satisfiability was solved in a seminal paper by Schaefer [27].

The most well studied variant of the minimal inference problem and also the most difficult variant to analyse is the classical one where we require all variables to be minimized, i.e., with  $Z = \emptyset$  and  $Q = \emptyset$ . This inference problem was proved  $\Pi_2\text{P}$ -complete by Eiter and Gottlob in [12] if no restrictions are imposed on the propositional theory  $\varphi$ . Cadoli and Lenzerini proved in [6] that the inference problem becomes  $\text{coNP}$ -complete if  $\varphi$  is a bijunctive or a dual Horn formula. Durand and Hermann proved in [11] that also the inference problem for affine formulas  $\varphi$  is  $\text{coNP}$ -complete. Finally, Kirousis and Kolaitis [18] proved a dichotomy theorem separating the  $\Pi_2\text{P}$ -complete cases from the cases in  $\text{coNP}$ . Moreover, they conjectured that the cases in  $\text{coNP}$  could be separated into  $\text{coNP}$ -complete cases and tractable (i.e., polynomial-time decidable) cases.

The proof of the dichotomy theorem for the minimal inference problem by Kirousis and Kolaitis [18] turned out to be difficult for several reasons. One of them was the impossibility to apply the well-known approach through the theory of clones and Post's lattice [3, 15, 25] to obtain a complexity classification. The culprit is the existential quantification which does not combine well with minimality. This implies that the co-clones, which are the sets of relations closed under variable identification, variable permutation, conjunction, and existential quantification, are not invariant with respect to the complexity of minimal inference. This situation precludes the use of the Galois correspondence, which allows first to prove a completeness result for a subset of relations and subsequently to extend it to the whole co-clone.

However, if the pointwise partial order on models is relaxed, which is the consequence of studying the generalized minimal inference problems where some variables are allowed to vary, then it has been observed that the Galois correspondence can be restored [21] and the approach via clone

theory works. With the help of these powerful tools it is relatively easy to classify the complexity of these generalized forms of minimal inference for every restriction on the types of allowed clauses in the theory (what we do in Sections 5 and 6).

As already mentioned, following the results of Kirousis and Kolaitis [18], it is clear that we cannot use the powerful algebraic approach for attacking the complexity of the minimal inference problem. Our approach is instead based on refinements of Schaefer's approach (via a particular type of *implementations*) to classify the complexity of the satisfiability problem in propositional logic (see [8, 27]). There is one significant difference though between Schaefer's implementations and ours. We can only use conjunction, variable identification, and variable permutation (*without* existential quantifiers) in our implementations. This is due to the fact, as already explained, that existential quantification does not preserve the minimal models, nor the complexity of the problem. Nevertheless, we manage to separate the coNP-hard cases from the tractable ones by using this approach through a more fine-grained analysis, thus, finishing the complexity classification for all four variants of minimal inference with all possible restrictions on the types of allowed clauses, establishing trichotomy theorems for all considered variants of the minimal inference problem.

Our paper finally settles with a positive answer the trichotomy conjecture of Kirousis and Kolaitis [18] for the basic variant of minimal inference, which was also listed as one of the open problems (Question 4.1) during the International Workshop on Mathematics of Constraint Satisfaction held in Oxford in March 2006 [1].

## 2 Preliminaries

Throughout the paper we use the standard correspondence between constraints and relations. We use the same symbol for a constraint and its corresponding relation, since the meaning will always be clear from the context, and we say that the constraint *represents* the relation.

An  $n$ -ary *logical relation*  $R$  is a Boolean relation of arity  $n$ . Each element of a logical relation  $R$  is an  $n$ -ary Boolean vector (also called a tuple)  $m = (m_1, \dots, m_n) \in \{0, 1\}^n$ . To save space, we will often write the vector  $(m_1, \dots, m_n)$  in the form  $m_1 \cdots m_n$ . Let  $V$  be a set of variables. A *constraint* is an application of  $R$  to an  $n$ -tuple of variables from  $V$ , i.e.,  $R(x_1, \dots, x_n)$ .

Consider a relation  $R$  represented by a Boolean matrix, i.e., the vectors of  $R$  constitute the rows of the matrix. We say that a relation  $R$  is *irredundant* if it does not contain two identical columns and it cannot be transformed by column permutation to a relation of the form  $Q \times \{0, 1\}^k$  for a  $k \geq 1$ , where  $Q$  is another relation. If  $R$  is redundant then the corresponding *irredundant reduction*  $R^\circ$  is formed by identifying identical columns and removing all columns of the form  $\{0, 1\}^k$ . A set of relations  $S$  is irredundant if every relation in  $S$  is irredundant. Given a set of relations  $S$ , we form the irredundant reduction  $S^\circ$  by replacing all redundant relations  $R$  in  $S$  by their corresponding irredundant reductions  $R^\circ$ .

Let  $R \subseteq \{0, 1\}^k$  be a  $k$ -ary Boolean relation. We say that a relation  $R'$  is a *direct 0-section* of  $R$  if there exists an index  $i \in \{1, \dots, k\}$ , such that

$$R' = \{(m[1], \dots, m[i-1], m[i+1], \dots, m[k]) \mid m \in R \wedge m[i] = 0\}.$$

We say that a relation  $R''$  is a *0-section* of  $R$  if there exists a finite sequence of Boolean relations  $R_0, R_1, \dots, R_n$ , such that  $R_0 = R$ ,  $R'' = R_n$ , and  $R_{j+1}$  is a direct 0-section of  $R_j$  for each  $j = 0, \dots, n-1$ . Let  $S$  be a finite set of Boolean relations. We say that  $S^*$  is a 1-valid *restriction* of  $S$  if it contains all relations  $R^*$  which are both 1-valid and a 0-section of a relation from  $S$ .

Note that starting from an arbitrary relation  $R \in S$ , we always arrive at a 1-valid 0-section  $R^*$  by iterating the 0-section operation long enough, unless  $R = \{0 \cdots 0\}$ . The aforementioned technique is a well-known concept from coding theory and it was already used for circumscription [17] and minimal inference [18].

An *assignment* is a mapping  $m: V \rightarrow \{0, 1\}$  assigning a Boolean value  $m(x)$  to each variable  $x \in V$ . If we arrange the variables in some arbitrary but fixed order, say as a vector  $(x_1, \dots, x_n)$ , then the assignments can be identified with vectors from  $\{0, 1\}^n$ . The  $i$ -th component of a vector  $m$ , denoted by  $m[i]$ , corresponds to the value of the  $i$ -th variable, i.e.,  $m(x_i) = m[i]$ . An assignment  $m$  satisfies the constraint  $R(x_1, \dots, x_n)$  if  $(m(x_1), \dots, m(x_n)) \in R$  holds. An assignment  $m$  satisfying a constraint  $R$  is called a *model* of  $R$ .

Throughout the text we refer to different types of Boolean constraint relations following Schaefer's terminology [27] (see also the monograph [8]). We say that a Boolean relation  $R$  is

- *1-valid* if  $1 \cdots 1 \in R$  and it is *0-valid* if  $0 \cdots 0 \in R$ ,
- *Horn* (*dual Horn*) if  $R$  can be represented by a conjunctive normal form (CNF) formula having at most one unnegated (negated) variable in each clause,
- *bijunctive* if it can be represented by a CNF formula having at most two variables in each clause,
- *affine* if it can be represented by an affine system of equations  $S: Ax = b$  over  $\mathbb{Z}_2$ ,
- *binary affine*, sometimes also called width-2 affine, if it is both affine and bijunctive,
- *complementive* if for each  $(m_1, \dots, m_n) \in R$  also  $(\neg m_1, \dots, \neg m_n) \in R$ .

A set  $S$  of Boolean relations is called 0-valid (1-valid, Horn, dual Horn, affine, bijunctive, complementive) if *every* relation in  $S$  is 0-valid (1-valid, Horn, dual Horn, affine, bijunctive, complementive). A relation  $R$  (a set of relations  $S$ ) is called *Schaefer* if it belongs to one of the classes Horn, dual Horn, bijunctive, or affine. Note that if  $S$  is Schaefer, then so is  $S^*$ . This is because the Schaefer classes are stable under constant substitution. Also note that  $S$  is 0-valid if and only if  $S^*$  is 0-valid.

Let  $S$  be a non-empty finite set of Boolean relations, also called a *constraint language*. An *S-formula* is a finite conjunction of clauses  $\varphi = c_1 \wedge \cdots \wedge c_k$ , where each clause  $c_i$  is a constraint application of some logical relation  $R \in S$ . An assignment  $m$  satisfies the formula  $\varphi$  if it satisfies all clauses  $c_i$ . Hence the notion of models naturally extends from constraints to formulas. We denote by  $[\varphi]$  the set of models of a formula  $\varphi$ . It is clear that each  $[\varphi]$  denotes a Boolean relation. In this case we also say that the formula (constraint)  $\varphi$  *generates* the relation  $[\varphi]$ .

Let  $m = (m[1], \dots, m[n])$  and  $m' = (m'[1], \dots, m'[n])$  be two Boolean vectors from  $\{0, 1\}^n$ . We write  $m \leq m'$  to denote that  $m[i] \leq m'[i]$  holds for every  $i \leq n$ , as well as  $m < m'$  for  $m \leq m'$  and  $m \neq m'$ . The relation  $\leq$  is called the *pointwise partial order* on models. Let  $\varphi(x_1, \dots, x_n)$  be a Boolean formula having  $x_1, \dots, x_n$  as its variables and let  $m \in \{0, 1\}^n$  be a truth assignment. We say that  $m$  is a *minimal model* of  $\varphi$  if  $m$  is a model of  $\varphi$  and there is no other model  $m'$  of  $\varphi$  that satisfies the relation  $m' < m$ . Two vectors  $a, b \in R$  satisfying the relation  $a < b$  are called *comparable*. We say that a relation  $R$  is *incomparable* if it does not contain comparable vectors. We say that a set of relations  $S$  is incomparable if each relation  $R \in S$  is incomparable.

Let us partition the variables  $V$  into three disjoint sets  $P$ ,  $Z$ , and  $Q$ , as it is usual in minimality problems in artificial intelligence (see [6, 14]), where  $P$  is the set of variables to be minimized,  $Z$  is the set of variables allowed to vary (also called free variables), and  $Q$  is the set of variables that must maintain a constant value (also called constants). We write  $m \prec m'$  to denote that the Boolean vectors  $m$  and  $m'$  satisfy the following conditions: (1)  $m(x) < m'(x)$  for all  $x \in P$  and

(2)  $m(x) = m'(x)$  for all variables  $x \in Q$ . We write  $m \preceq m'$  if the first condition is  $m \leq m'$ . Note that there is no condition on the values of variables in  $Z$ . We say that  $m$  is a  $(P, Z, Q)$ -*minimal* or *generalized minimal model* of  $\varphi$  if  $m$  is a satisfying truth assignment of  $\varphi$  and there is no other satisfying truth assignment  $m'$  of  $\varphi$  that satisfies the relation  $m' \prec m$ . If the variable sets  $Z$  and  $Q$  are empty, then we identify the  $(P, \emptyset, \emptyset)$ -minimal models with the aforementioned minimal models of  $\varphi$ .

Let  $\varphi$  and  $\psi$  be two propositional formulas in conjunctive normal form. We say that  $\psi$  follows from  $\varphi$  in propositional circumscription, denoted by  $\varphi \models_{\min} \psi$ , if  $\psi$  is true in every minimal model of  $\varphi$ . Since  $\psi$  is a conjunction  $c_1 \wedge \dots \wedge c_k$  of clauses  $c_i$ , then  $\varphi \models_{\min} \psi$  if and only if  $\varphi \models_{\min} c_i$  for each  $i$ . Hence we can restrict ourselves to consider only a single clause instead of a formula  $\psi$  at the right-hand side of the propositional inference problem  $\varphi \models_{\min} c$ . Similarly, we say that  $\psi$  follows from  $\varphi$  in  $(P, Z, Q)$ -circumscription, also denoted by  $\varphi \models_{\min}^{Z, Q} \psi$ , if  $\psi$  is true in every  $(P, Z, Q)$ -minimal model of  $\varphi$ . The superscripts  $Z$  and  $Q$  are usually dropped from  $\varphi \models_{\min}^{Z, Q} \psi$  when their presence is clear from the context.

### 3 Clones, Co-Clones, and Post's Lattice

The theory of clones and co-clones have previously shown to be very useful for characterizing the complexity of problems with restrictions on the types of allowed constraints (see for instance the survey [2,3]). The underlying theory, a part of universal algebra, can be found in the monograph [24] or in Chapter 1 of the book [23].

The theory of Boolean clones studies the closure properties of Boolean relations with respect to Boolean functions. The notion of closure property of a Boolean relation has been defined in a general way. Let  $f: \{0, 1\}^k \rightarrow \{0, 1\}$  be a Boolean function of arity  $k$ . We say that  $R$  is *closed* under  $f$ , or that  $f$  is a *polymorphism* of  $R$ , if for any choice of  $k$  vectors  $m_1, \dots, m_k \in R$ , we have that

$$\left( f(m_1[1], \dots, m_k[1]), f(m_1[2], \dots, m_k[2]), \dots, f(m_1[n], \dots, m_k[n]) \right) \in R,$$

i.e., that the new vector constructed coordinate-wise from  $m_1, \dots, m_k$  by means of  $f$  belongs to  $R$ .

We denote by  $\text{Pol}(R)$  the set of all polymorphisms of  $R$  and by  $\text{Pol}(S)$  the set of Boolean functions that are polymorphisms of every relation in  $S$ . It turns out that  $\text{Pol}(S)$  is a *closed set of Boolean functions*, also called a *clone*, for every set of relations  $S$ . A *clone* is a set of functions containing all projections and closed under composition. In particular, the set of all function  $[F]$  that can be constructed from a given functions  $F$  by means of composition, variable identification, and variable permutation, forms a clone.

We denote by  $\text{Inv}(F)$  the set of relations *invariant* under the functions  $F$ . In other words, for each relation  $R \in \text{Inv}(F)$  and each Boolean function  $f \in F$  we have that  $f$  is a polymorphism of  $R$ . It turns out that  $\text{Inv}(F)$  is a *closed set of Boolean relations*, also called a *co-clone*, for every set of functions  $F$ . Given a set of functions  $F$  we denote the corresponding co-clone  $\text{Inv}(F)$  by  $iF$ .

We denote by  $\langle S \rangle$  the *co-clone* generated by the set of relations  $S$ , also called the *co-clone closure* of  $S$ , i.e., the smallest set containing  $S$ , the binary equality relation  $eq^2$ , and closed under Cartesian product, permutation, restriction, and projection. The co-clone  $\langle S \rangle$  can also be understood as the set of all models of formulas built from constraints over  $S$ , and closed under conjunction, variable permutation, variable identification, and existential quantification. Keeping in mind that  $[\varphi]$  is the set of model of  $\varphi$ , this means that  $\langle S \rangle$  can be defined inductively as follows:

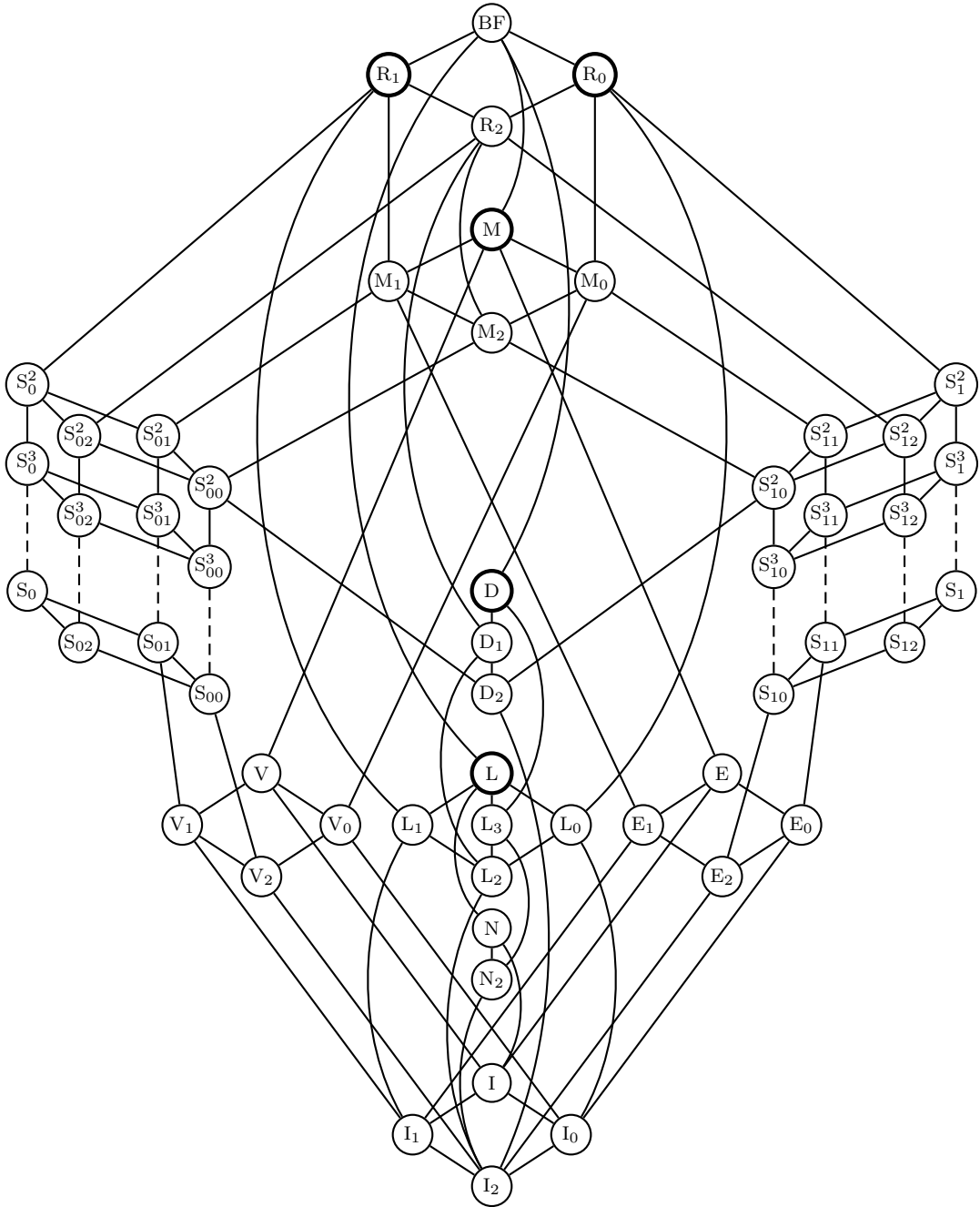


Figure 1: Post's lattice of Boolean clones

Clone	Base(s)	Co-clone	Base(s)
BF	$\{x \wedge y, \neg x\}$	$i$ BF	$\{eq^2\}$
R <sub>0</sub>	$\{x \wedge y, x \oplus y\}$	$i$ R <sub>0</sub>	$\{\neg x\}$
R <sub>1</sub>	$\{x \vee y, x \oplus y \oplus 1\}$	$i$ R <sub>1</sub>	$\{x\}$
R <sub>2</sub>	$\{x \vee y, x \wedge (y \oplus z \oplus 1)\}$	$i$ R <sub>2</sub>	$\{\neg x, x\}$
M	$\{x \wedge y, x \vee y, 0, 1\}$	$i$ M	$\{x \rightarrow y\}$
M <sub>1</sub>	$\{x \wedge y, x \vee y, 1\}$	$i$ M <sub>1</sub>	$\{x \rightarrow y, x\}$
M <sub>0</sub>	$\{x \wedge y, x \vee y, 0\}$	$i$ M <sub>0</sub>	$\{x \rightarrow y, \neg x\}$
M <sub>2</sub>	$\{x \wedge y, x \vee y\}$	$i$ M <sub>2</sub>	$\{x \rightarrow y, \neg x, x\}$
S <sub>0</sub> <sup>m</sup>	$\{x \rightarrow y, \text{dual}(h_m)\}$	$i$ S <sub>0</sub> <sup>m</sup>	$\{or^m\}$
S <sub>1</sub> <sup>m</sup>	$\{x \wedge \neg y, h_m\}$	$i$ S <sub>1</sub> <sup>m</sup>	$\{nand^m\}$
S <sub>0</sub>	$\{x \rightarrow y\}$	$i$ S <sub>0</sub>	$\{or^m \mid m \geq 2\}$
S <sub>1</sub>	$\{x \wedge \neg y\}$	$i$ S <sub>1</sub>	$\{nand^m \mid m \geq 2\}$
S <sub>02</sub> <sup>m</sup>	$\{x \vee (y \wedge \neg z)\}$	$i$ S <sub>02</sub> <sup>m</sup>	$\{or^m, \neg x, x\}$
S <sub>02</sub>	$\{x \vee (y \wedge \neg z)\}$	$i$ S <sub>02</sub>	$\{or^m \mid m \geq 2\} \cup \{\neg x, x\}$
S <sub>01</sub> <sup>m</sup>	$\{\text{dual}(h_m), 1\}$	$i$ S <sub>01</sub> <sup>m</sup>	$\{or^m, x \rightarrow y\}$
S <sub>01</sub>	$\{x \vee (y \wedge z), 1\}$	$i$ S <sub>01</sub>	$\{or^m \mid m \geq 2\} \cup \{x \rightarrow y\}$
S <sub>00</sub> <sup>m</sup>	$\{x \vee (y \wedge z), \text{dual}(h_m)\}$	$i$ S <sub>00</sub> <sup>m</sup>	$\{or^m, x \rightarrow y, \neg x, x\}$
S <sub>00</sub>	$\{x \vee (y \wedge z)\}$	$i$ S <sub>00</sub>	$\{or^m \mid m \geq 2\} \cup \{\neg x, x, x \rightarrow y\}$
S <sub>12</sub> <sup>m</sup>	$\{x \wedge (y \vee \neg z), \text{dual}(h_m)\}$	$i$ S <sub>12</sub> <sup>m</sup>	$\{nand^m, \neg x, x\}$
S <sub>12</sub>	$\{x \wedge (y \vee \neg z)\}$	$i$ S <sub>12</sub>	$\{nand^m \mid m \geq 2\} \cup \{\neg x, x\}$
S <sub>11</sub> <sup>m</sup>	$\{h_m, 0\}$	$i$ S <sub>11</sub> <sup>m</sup>	$\{nand^m, x \rightarrow y\}$
S <sub>11</sub>	$\{x \wedge (y \vee z), 0\}$	$i$ S <sub>11</sub>	$\{nand^m \mid m \geq 2\} \cup \{x \rightarrow y\}$
S <sub>10</sub> <sup>m</sup>	$\{x \wedge (y \vee z), h_m\}$	$i$ S <sub>10</sub> <sup>m</sup>	$\{nand^m, \neg x, x, x \rightarrow y\}$
S <sub>10</sub>	$\{x \wedge (y \vee z)\}$	$i$ S <sub>10</sub>	$\{nand^m \mid m \geq 2\} \cup \{\neg x, x, x \rightarrow y\}$
D	$\{(x \wedge \neg y) \vee (x \wedge \neg z) \vee (\neg y \wedge \neg z)\}$	$i$ D	$\{x \oplus y\}$
D <sub>1</sub>	$\{(x \wedge y) \vee (x \wedge \neg z) \vee (y \wedge \neg z)\}$	$i$ D <sub>1</sub>	$\{x \oplus y, x\}$
D <sub>2</sub>	$\{(x \vee y) \wedge (y \vee z) \wedge (z \vee x)\}$	$i$ D <sub>2</sub>	$\{x \oplus y, x \rightarrow y\}$
L	$\{x \oplus y, 1\}$	$i$ L	$\{\text{even}^4\}$
L <sub>0</sub>	$\{x \oplus y\}$	$i$ L <sub>0</sub>	$\{\text{even}^4, \neg x\}, \{\text{even}^3\}$
L <sub>1</sub>	$\{x \equiv y\}$	$i$ L <sub>1</sub>	$\{\text{even}^4, x\}, \{\text{odd}^3\}$
L <sub>2</sub>	$\{x \oplus y \oplus z\}$	$i$ L <sub>2</sub>	$\{\text{even}^4, \neg x, x\}$
L <sub>3</sub>	$\{x \oplus y \oplus z \oplus 1\}$	$i$ L <sub>3</sub>	$\{\text{even}^4, x \oplus y\}$
V	$\{x \vee y, 0, 1\}$	$i$ V	$\{x \vee y \vee \neg z\}$
V <sub>0</sub>	$\{x \vee y, 0\}$	$i$ V <sub>0</sub>	$\{x \vee y \vee \neg z, \neg x\}$
V <sub>1</sub>	$\{x \vee y, 1\}$	$i$ V <sub>1</sub>	$\{x \vee y \vee \neg z, x\}$
V <sub>2</sub>	$\{x \vee y\}$	$i$ V <sub>2</sub>	$\{x \vee y \vee \neg z, \neg x, x\}$
E	$\{x \wedge y, 0, 1\}$	$i$ E	$\{\neg x \vee \neg y \vee z\}$
E <sub>0</sub>	$\{x \wedge y, 0\}$	$i$ E <sub>0</sub>	$\{\neg x \vee \neg y \vee z, \neg x\}$
E <sub>1</sub>	$\{x \wedge y, 1\}$	$i$ E <sub>1</sub>	$\{\neg x \vee \neg y \vee z, x\}$
E <sub>2</sub>	$\{x \wedge y\}$	$i$ E <sub>2</sub>	$\{\neg x \vee \neg y \vee z, \neg x, x\}$
N	$\{\neg x, 0\}, \{\neg x, 1\}$	$i$ N	$\{\text{dup}^3\}$
N <sub>2</sub>	$\{\neg x\}$	$i$ N <sub>2</sub>	$\{\text{nae}^3\}$
I	$\{\text{id}, 0, 1\}$	$i$ I	$\{\text{even}^4, x \rightarrow y\}$
I <sub>0</sub>	$\{\text{id}, 0\}$	$i$ I <sub>0</sub>	$\{\text{even}^4, x \rightarrow y, \neg x\}, \{\text{dup}^3, x \rightarrow y\}$
I <sub>1</sub>	$\{\text{id}, 1\}$	$i$ I <sub>1</sub>	$\{\text{even}^4, x \rightarrow y, x\}$
I <sub>2</sub>	$\{\text{id}\}$	BR	$\{1\text{-in-3}\}$

Table 1: List of Boolean clones and co-clones with bases

Functions	$h_m = \bigvee_{i=1}^{m+1} x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_{m+1}$ $\text{dual}(h_m)(x_1, \dots, x_{m+1}) = \neg h_m(\neg x_1, \dots, \neg x_{m+1})$ $(x \oplus y) = x + y \pmod{2}$ $(x \equiv y) = x + y + 1 \pmod{2}$ $\text{id}(x) = x$
Relations	$\text{or}^m = \{(a_1, \dots, a_m) \in \{0, 1\}^m \mid \sum_{i=1}^m a_i > 0\}$ $\text{nand}^m = \{(a_1, \dots, a_m) \in \{0, 1\}^m \mid \sum_{i=1}^m a_i \neq m\}$ $\text{even}^m = \{(a_1, \dots, a_m) \in \{0, 1\}^m \mid \sum_{i=1}^m a_i \text{ is even}\}$ $\text{odd}^m = \{(a_1, \dots, a_m) \in \{0, 1\}^m \mid \sum_{i=1}^m a_i \text{ is odd}\}$ $\text{dup}^3 = \{0, 1\}^3 \setminus \{010, 101\}$ $\text{nae}^3 = \{0, 1\}^3 \setminus \{000, 111\}$ $\text{eq}^2 = \{00, 11\}$ $\text{1-in-3} = \{001, 010, 100\}$

Table 2: List of Boolean functions and relations

1.  $S \subseteq \langle S \rangle$  and  $\text{eq}^2 \in \langle S \rangle$ ,
2. if  $R_1, R_2 \in \langle S \rangle$  then  $R_1 \times R_2 = [R_1(\vec{x}) \wedge R_2(\vec{y})] \in \langle S \rangle$ , (Cartesian product or conjunction)
3. if  $[R(\vec{x}, u, \vec{y}, v, \vec{y})] \in \langle S \rangle$  then  $[R(\vec{x}, v, \vec{y}, u, \vec{y})] \in \langle S \rangle$ , (permutation)
4. if  $R(x_1, x_2, \vec{y}) \in \langle S \rangle$  then  $[R(x, x, \vec{y})] \in \langle S \rangle$ , where  $x$  is a fresh variable, (restriction)
5. if  $[R(x, \vec{y})] \in \langle S \rangle$  then  $[\exists x R(x, \vec{y})] \in \langle S \rangle$ . (projection or existential quantification)

We will also require a weaker closure than  $\langle S \rangle$ , called *weak co-clone closure* and denoted by  $\langle S \rangle_{\#}$ , where the set is closed under the first four operations, but *not* under projection (existential quantification) depicted in Condition 5. The *idempotent closure*  $S^{\text{id}}$  of a set of relations  $S$  corresponds to  $S \cup \{ \{0\}, \{1\} \}$ , which can be also written as  $S \cup \{ [x], [\neg x] \}$ .

A *Galois correspondence* has been exhibited between the sets of Boolean functions  $\text{Pol}(S)$  and the sets of Boolean relations  $S$ , as well as between the functions  $F$  and the relations  $\text{Inv}(F)$ . The Galois correspondence encompass the following implications. Let  $F_1$  and  $F_2$  be two sets of Boolean functions, as well as  $S_1$  and  $S_2$  two sets of Boolean relations. The inclusion  $[F_1] \subseteq [F_2]$  implies  $\text{Inv}(F_1) \supseteq \text{Inv}(F_2)$  and the inclusion  $\langle S_1 \rangle \subseteq \langle S_2 \rangle$  implies  $\text{Pol}(S_1) \supseteq \text{Pol}(S_2)$ . In particular, this Galois correspondence enables the proof of the identities  $\text{Pol}(\text{Inv}(F)) = [F]$  and  $\text{Inv}(\text{Pol}(S)) = \langle S \rangle$  for every set of functions  $F$  and each set of relations  $S$ . Moreover, it helps to easily establish complexity results for problems compatible with co-clones.

All clones of Boolean functions were identified by Post in [25]. Post also detected the inclusion structure of these clones, what is now referred to as *Post's lattice* (see Figure 1), as well as the property of each Boolean clone to have a finite *basis*, a smallest finite set of Boolean functions  $F$  identifying the clone  $[F]$ . The list of all clones and co-clones, together with their bases, according to Böhler *et al.* [2–4], can be found in Table 1, with the used functions and relations presented in Table 2. In the last column of Table 1 we often write only the constraints  $\varphi$  instead of the generated relation  $[\varphi]$ .

In the sequel we need to determine the properties of relations. This will be done by means of closure operations of the corresponding clones. In particular, we study the closure under the Boolean functions called conjunction ( $\wedge$ ), disjunction ( $\vee$ ), majority ( $\text{maj}$ ), addition over the Boolean ring  $\mathbb{Z}_2$

$R$ is Horn	$\Leftrightarrow$	$m, m' \in R$ implies $m \wedge m' \in R$
$R$ is dual Horn	$\Leftrightarrow$	$m, m' \in R$ implies $m \vee m' \in R$
$R$ is bijunctive	$\Leftrightarrow$	$m, m', m'' \in R$ implies $\text{maj}(m, m', m'') \in R$
$R$ is affine	$\Leftrightarrow$	$m, m', m'' \in R$ implies $m + m' + m'' \in R$
$R$ is complementive	$\Leftrightarrow$	$m \in R$ implies $\neg m \in R$

Table 3: Correspondence with closure operations

called exclusive-or (+), and negation ( $\neg$ ), where the majority operation is equivalent to the identity  $\text{maj}(m, m', m'') = (m \vee m') \wedge (m' \vee m'') \wedge (m'' \vee m)$ . These operations are applied componentwise. The correspondence between the type of relation and the closure operation is specified in Table 3.

## 4 Generalized Minimal Inference and Extension

We are interested in the complexity of the following problem, which is a generalization of the minimal inference problem studied in [12, 18]. See, e.g., the paper [16] for an overview of existing complexity results for corresponding circumscription and minimal inference problems.

**Problem:**  $\text{GMININF}(S)$

*Input:* A set of variables  $V$  divided into three disjoint sets  $P$ ,  $Z$ , and  $Q$ , an  $S$ -formula  $\varphi$  over the variables  $V$ , and a clause  $\psi$ .

*Question:* Is  $\psi$  satisfiable in every  $(P, Z, Q)$ -minimal model of  $\varphi$ , i.e., does  $\varphi \models_{\min} \psi$  hold?

According to the (non)emptiness of the sets  $Z$  and  $Q$ , there exist four variants of the generalized minimal inference problem, namely

$\text{GMININF}$	if $Z \neq \emptyset$ and $Q \neq \emptyset$ ,	$\text{VMININF}$	if $Z \neq \emptyset$ and $Q = \emptyset$ ,
$\text{CMININF}$	if $Z = \emptyset$ and $Q \neq \emptyset$ ,	$\text{MININF}$	if $Z = Q = \emptyset$ .

We will call  $\text{VMININF}$  the minimal inference with free variables and  $\text{CMININF}$  the minimal inference with constants.  $\text{MININF}$  is the classical inference problem where all variables are minimized, as it was considered in [12, 18].

For the non-parametrized case, when we consider the class of formulas as a whole and not produced from a given set of relations  $S$ , the following results have been previously proved. Eiter and Gottlob proved in [12] that  $\text{MININF}$  is  $\Pi_2\text{P}$ -complete. Cadoli and Lenzerini [6] showed that the  $\text{MININF}$  problem for dual Horn or bijunctive formulas  $\varphi$  is  $\text{coNP}$ -complete. Durand and Hermann [11] showed that the  $\text{MININF}$  problem for affine formula  $\varphi$  is  $\text{coNP}$ -complete. For the parametrized case Kirousis and Kolaitis [18] showed that there exists a dichotomy between the  $\Pi_2\text{P}$ -complete case of  $\text{MININF}(S)$  for a the whole set of relations  $S$  and the special cases of  $S$  for which  $\text{MININF}(S)$  included in  $\text{coNP}$ . The  $\text{MININF}(S)$  problem for a set of Horn relations  $S$  is trivially known to be polynomial-time decidable, since a Horn formula is known to have at most one minimal model, depending on its satisfiability.

The  $\text{GMININF}(S)$  problem and its complexity can be naturally extended to infinite sets of relations  $S$  in the following way.  $\text{GMININF}(S)$  is decidable in polynomial time for an infinite set  $S$  if for each finite set of relations  $S' \subseteq S$  the problem  $\text{GMININF}(S')$  is decidable in polynomial time.  $\text{GMININF}(S)$  is  $\text{coNP}$ -complete if there exists a finite subset  $S' \subseteq S$ , such that  $\text{GMININF}(S')$  is

coNP-complete and there is no finite subset  $S'' \subseteq S$  for which  $\text{GMININF}(S'')$  is  $\Pi_2\text{P}$ -hard. Finally,  $\text{GMININF}(S)$  is  $\Pi_2\text{P}$ -complete if there exists a finite subset  $S' \subseteq S$ , such that  $\text{GMININF}(S')$  is  $\Pi_2\text{P}$ -complete. We will need these notions when we will speak about the problem  $\text{GMININF}(\langle S \rangle)$ .

It should be clear that membership results propagate from more general to more restrictive variants of the minimal inference problem. In the same spirit, hardness results propagate from more restrictive to more general variants of minimal inference. Hence, a polynomial-time decidability of  $\text{GMININF}$  propagates to  $\text{VMININF}$ ,  $\text{CMININF}$ , and  $\text{MININF}$ , whereas a hardness result for  $\text{MININF}$  propagates to the other three variants.

Nordh and Jonsson [21] observed that the Galois correspondence exists for the  $\text{GMININF}$  problem, what allowed them to prove the following results. The proof in [21] is done only for  $\text{GMININF}$ , but the same technique carries over also to  $\text{VMININF}$ , since the proof is based on a construction concerning only variables.

**Proposition 4.1** ([21]) *If  $\langle S_1 \rangle \subseteq \langle S_2 \rangle$  then  $\text{GMININF}(S_1) \leq_m^p \text{GMININF}(S_2)$  and  $\text{VMININF}(S_1) \leq_m^p \text{VMININF}(S_2)$ .*

**Theorem 4.2** ([21]) *The problems  $\text{GMININF}(S)$  and  $\text{GMININF}(\langle S \rangle)$  are polynomially equivalent for a set of Boolean relations  $S$ .*

As Nordh and Jonsson showed, the proof of this theorem is made possible because the existentially quantified variables  $x_{p+1}, \dots, x_q$  of a constructed relation

$$R(x_1, \dots, x_p) = \exists x_{p+1} \cdots \exists x_q R_1(x_1, \dots, x_q) \wedge \cdots \wedge R_r(x_1, \dots, x_q)$$

from a given set of relations  $S = \{R_1, \dots, R_r\}$ , can be added to the set of varying variables  $Z$ . Hence, the generalized minimal inference problem for a given set of relations  $S$  and its co-clone closure  $\langle S \rangle$  has the same complexity. This result remains true also for the  $\text{VMININF}$  problem. However, there is no known way to derive an equivalent to Theorem 4.2 for the  $\text{CMININF}$  and  $\text{MININF}$  problems, mainly because existentially quantified variables cannot be handled in the appropriate way and therefore the universal algebra approach cannot be applied.

Sometimes it is convenient to consider the idempotent closure  $S^{\text{id}}$  instead of the original set of relations  $S$ . However, with respect to the problems  $\text{GMININF}$  and  $\text{CMININF}$  there is no difference between  $S$  and  $S^{\text{id}}$  from the complexity point of view, as it was observed by Nordh and Jonsson [21].

**Theorem 4.3** ([21]) *The problems  $\text{GMININF}(S)$  and  $\text{GMININF}(S^{\text{id}})$ , as well as  $\text{CMININF}(S)$  and  $\text{CMININF}(S^{\text{id}})$  are polynomially equivalent for any set of Boolean relations  $S$ .*

As Durand and Hermann showed in [11], it is sometimes more convenient to investigate the dual problem  $\text{GMINEXT}$  of generalized minimal extension, defined as follows.

**Problem:**  $\text{GMINEXT}(S)$

*Input:* A set of variables  $V$  divided into three disjoint sets  $P$ ,  $Z$ , and  $Q$ , an  $S$ -formula  $\varphi$  over variables  $V$ , and a partial assignment  $m$  of  $\varphi$  for a subset of variables  $V' \subseteq V$ .

*Question:* Can  $m$  be extended to a  $(P, Z, Q)$ -minimal model  $\bar{m}$  of  $\varphi$ ?

The problems  $\text{VMINEXT}$ ,  $\text{CMINEXT}$ , and  $\text{MINEXT}$  are derived from  $\text{GMINEXT}$  by the corresponding conditions on the sets  $Z$  and  $Q$ , as in the case of the  $\text{GMININF}$  problem. If  $m$  can be extended to a  $(P, Z, Q)$ -minimal model  $\bar{m}$  of  $\varphi$ , but the model  $\bar{m}$  is not specified, we also say that  $m$  has

a  $(P, Z, Q)$ -minimal extension. The relationship between GMININF and GMINEXT can be easily established through the following construction. Let  $S$  be a finite set of Boolean relations and  $m$  an assignment to the variables  $x_1, \dots, x_n$ . Let  $c_x$  be the largest clause falsified by the assignment  $m$ , i.e.,  $c_m = l_1 \vee \dots \vee l_n$ , where  $l_i = x_i$  if  $m(x_i) = 0$ , and  $l_i = \neg x_i$  otherwise. It is clear that the clause  $c_m$  is *not* satisfiable in every  $(P, Z, Q)$ -minimal model of the formula  $\varphi$  if and only if the assignment  $m$  can be extended to a  $(P, Z, Q)$ -minimal model  $\bar{m}$  of  $\varphi$ . From this follows that for each set of relations  $S$  the problem GMININF( $S$ ) is  $\Pi_2\text{P}$ -complete, coNP-complete, or polynomial-time decidable if and only if GMINEXT( $S$ ) is  $\Sigma_2\text{P}$ -complete, NP-complete, or polynomial-time decidable, respectively. This also holds for the problems VMININF and VMINEXT, as well as for MININF and MINEXT.

Following Nordh and Jonsson [21], the Galois correspondence holds for GMININF and VMININF. However, no such result is known for CMININF and MININF. In the latter cases we need to consider the weak co-clones  $\langle S \rangle_{\#}$  instead of the usual clones  $\langle S \rangle$ . To be able to perform the required complexity analysis, we need reduction theorems between minimal inference problems parametrized by different sets of relations. Since our sets of relations  $\langle S \rangle_{\#}$  are not closed under existential quantification, we cannot have the usual reduction theorem based on inclusion of polymorphisms.

**Proposition 4.4** *Let  $R$  be a Boolean relation and  $S$  a set of relations. If  $R \in \langle S \rangle_{\#}$  then there exists a polynomial many-one reduction from MININF( $R$ ) to MININF( $S$ ), as well as from CMININF( $R$ ) to CMININF( $S$ ).*

**Proof:** Let  $\varphi \models_{\min} \psi$  be an instance of MININF( $R$ ), where  $\varphi$  is a conjunction of constraints built upon the relation  $R$ . Since  $R \in \langle S \rangle_{\#}$  holds, every constraint  $R(x_1, \dots, x_k)$  can be written as a conjunction of constraints upon relations from  $S$ . Substitute the latter into  $\varphi$ , obtaining the new formula  $\varphi'$ . Now,  $\varphi' \models_{\min} \psi$  is an instance of MININF( $S$ ), where  $\varphi'$  is only polynomially larger than  $\varphi$ . It is clear that  $\varphi$  and  $\varphi'$  have the same variables and therefore also the same models. Hence, also the minimal models of  $\varphi$  and  $\varphi'$  are the same. The same reasoning also holds for CMININF.  $\square$

It is more convenient to work only with irredundant relations. It is intuitively clear that the complexity of GMININF( $S$ ) and GMININF( $S^\circ$ ) for a set of relations  $S$  and its irredundant reduction  $S^\circ$  is the same. Moreover, an equivalent result holds for the other three variants of the problem. For GMININF and VMININF it follows from the Galois correspondence. For CMININF and MININF we have the following result.

**Proposition 4.5** *If  $S^\circ$  is the irredundant reduction of a set of relations  $S$ , then MININF( $S$ ) and MININF( $S^\circ$ ), as well as CMININF( $S$ ) and CMININF( $S^\circ$ ), are equivalent under polynomial many-one reductions.*

**Proof:** Since the sets  $\langle S \rangle_{\#}$  and  $\langle S^\circ \rangle_{\#}$  are closed under variable permutation, the order of variables is irrelevant.

We perform the proof only for MININF, since the generalization to CMININF is the same. The reduction  $\text{MININF}(S^\circ) \leq_m^p \text{MININF}(S)$  is straightforward. If  $R^\circ \in S^\circ$  is constructed from  $R$  by identification of columns  $i, k + 1$  and  $\varphi^\circ$  contains the constraint  $R^\circ(x_1, \dots, x_k)$ , then  $\varphi$  is equal to  $\varphi^\circ$  in which we replace  $R^\circ(x_1, \dots, x_k)$  by  $R(x_1, \dots, x_i, \dots, x_k, x_i)$ . It is clear that  $\varphi^\circ$  and  $\varphi$  have the same models. Similarly, if  $R^\circ \in S^\circ$  is constructed from  $R = R^\circ \times \{0, 1\}^p$  for a  $p \geq 1$  and  $\varphi^\circ$  contains the constraint  $R^\circ(x_1, \dots, x_k)$ , then  $\varphi$  is equivalent to  $\varphi^\circ$  in which we replace  $R^\circ(x_1, \dots, x_k)$

by  $R(x_1, \dots, x_k, x_{k+1}, \dots, x_p)$ , where  $x_{k+1}, \dots, x_p$  are fresh variables. It is clear that  $m^\circ$  is a model of  $\varphi^\circ$  if and only if  $m^\circ \cdot m'$  is a model of  $\varphi$  for all  $m' \in \{0, 1\}^p$ . Hence, it is clear that  $m^\circ$  is a minimal model of  $\varphi^\circ$  if and only if  $m^\circ \cdot 0 \cdots 0$  with  $p$  trailing zeros is a minimal model of  $\varphi$ .

Conversely, let  $\varphi$  contain the constraint  $R(x_1, \dots, x_i, \dots, x_j, \dots, x_k)$  built upon  $R \in S$ . If the reduction  $R^\circ$  is constructed from  $R$  by identification of columns  $i$  and  $j$ , then we can replace in  $\varphi$  every occurrence of  $R(x_1, \dots, x_i, \dots, x_j, \dots, x_k)$  by  $R^\circ(x_1, \dots, x_i, \dots, x_{j-1}, x_{j+1}, \dots, x_k)$ , followed by the replacement of  $x_j$  by  $x_i$  in the rest of the formula  $\varphi$ . It is clear that  $m = (m[1], \dots, m[k])$  is a minimal model of  $R(x_1, \dots, x_k)$  if and only if  $m^\circ = (m[1], \dots, m[j-1], m[j+1], \dots, m[k])$  is a minimal model of  $R^\circ(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k)$ , since the relation  $R$  forces the coordinates  $i$  and  $j$  to be equal. Similarly, if  $R^\circ$  is constructed from  $R = R^\circ \times \{0, 1\}^p$  for a  $p \geq 1$  and  $\varphi$  contains the constraint  $R(x_1, \dots, x_k, \dots, x_{k+p})$ , then we can replace every occurrence of this constraint by  $R^\circ(x_1, \dots, x_k)$ . It is clear that  $m_0 = (m[1], \dots, m[k], 0, \dots, 0)$  is a minimal model of  $R(x_1, \dots, x_k, \dots, x_{k+p})$  if and only if  $m_0^\circ = (m[1], \dots, m[k])$  is a minimal model of  $R^\circ(x_1, \dots, x_k)$ .  $\square$

As a consequence, we assume throughout the paper that all constraint languages are irredundant.

Our goal is to derive a complete classification of complexity for the GMININF problem and its variants VMININF, CMININF, and MININF. Note that a membership result for GMININF carries over to all variants of minimal inference. In the same spirit, a  $\Pi_2\text{P}$ -hardness or  $\text{coNP}$ -hardness result for MININF also carries over to all variants.

## 5 Minimal Inference With Free Variables and Constants

We want to derive in this section a complete characterization of complexity for the GMININF problem. Theorem 4.2 is valid in this context, therefore we can use the algebraic approach based on clones and co-clones, exploiting the Galois correspondence. Moreover, Theorem 4.3 is valid and hence we can further restrict our attention to idempotent co-clones. Only one tractable case for the GMININF problem is known, namely when the constraint language  $S$  consists only of relations that are both Horn and dual Horn [6]. We prove two new tractability results for the GMININF problem for the classes of bijunctive affine relations and Horn relations containing only negative literals. These results together with the algebraic approach, combined with well-known complexity results from the literature are sufficient to give a complete complexity classification for the GMININF problem. We start with a relevant result from [21].

**Theorem 5.1** ([21]) *If the constraint language  $S$  is Horn, dual Horn, bijunctive, or affine, then  $\text{GMININF}(S)$  is in  $\text{coNP}$ . Otherwise,  $\text{GMININF}(S)$  is  $\Pi_2\text{P}$ -complete.*

Further we need to introduce some co-clones that will be of particular importance to us later.

**Co-clone  $i\mathbf{R}_2$ :** For  $a \in \{0, 1\}$ , a Boolean function  $f$  is called  $a$ -reproducing if  $f(a, \dots, a) = a$ . The clones  $\mathbf{R}_a$  contain all  $a$ -reproducing Boolean functions and the clone  $\mathbf{R}_2$  contains all functions that are both 0-reproducing and 1-reproducing. Hence  $i\mathbf{R}_2$  is the co-clone consisting of all relations closed under all functions that are both 0-reproducing and 1-reproducing. Note that functions satisfying  $f(a, \dots, a) = a$  for all  $a$  in its domain are usually called idempotent.

**Co-clone  $iD_1$ :** The co-clone  $iD_1$  consists of all relations closed under the operations  $\text{aff}(x, y, z) = x + y + z$  (affinity) and  $\text{maj}(x, y, z) = (x \vee y) \wedge (y \vee z) \wedge (x \vee z)$  (majority). Hence  $iD_1$  is the co-clone of all binary affine relations, which can be represented by systems of linear equations with at most two variables over the Boolean ring  $\mathbb{Z}_2$ .

**Co-clone  $iS_1$ :** The clone  $S_1$  consists of all 1-separating functions (see [2] for the definition). Dalmau proved in [9, Lemma 39] that  $iS_1$  is the co-clone consisting only of relations of the form  $\{0, 1\}^n \setminus (1, \dots, 1)$ . That is, the co-clone  $iS_1$  consists of all relations corresponding to conjunctions of clauses where all literals are negative, also called negative Horn.

As we have seen in Theorem 4.3, co-clones of the form  $\langle S^{\text{id}} \rangle$  are of particular importance to us. Remember that  $\text{GMININF}(S)$  is of the same complexity as  $\text{GMININF}(S^{\text{id}})$ . It is easy to see that a co-clone  $S$  is idempotent if and only if it is closed under all  $a$ -reproducing functions, i.e., that  $iR_2 \subseteq S$  holds. Hence we can derive the following lemma.

**Lemma 5.2** *Let  $S_1$  and  $S_2$  be co-clones such that  $S_2$  is the least upper bound of  $iR_2$  and  $S_1$  in Post's lattice of co-clones. Then  $\text{GMININF}(S_1)$  and  $\text{GMININF}(S_2)$  are polynomially equivalent.*

**Proof:** Remember that  $iR_2$  is the co-clone of all relations closed under all 0- and 1-reproducing functions. Thus,  $\{\{(0)\}, \{(1)\}\} \subseteq iR_2$ . If the set  $F$  contains a non-reproducing function  $f$ , then  $\{\{(0)\}, \{(1)\}\} \not\subseteq \text{Inv}(F)$ . Therefore given a co-clone  $S$ , we have  $\{\{(0)\}, \{(1)\}\} \subseteq S$  if and only if  $iR_2 \subseteq S$ . It follows that the least upper bound of  $iR_2$  and  $S_1$  in the lattice of co-clones is  $\langle S_1^{\text{id}} \rangle = S_2$ . Using Theorem 4.3 we get that  $\text{GMININF}(S_1)$  and  $\text{GMININF}(S_2)$  are polynomially equivalent.  $\square$

Now we present our new tractable cases for the GMININF problem. The first is the bijunctive affine case.

**Proposition 5.3**  *$\text{GMININF}(iD_1)$  is decidable in polynomial time.*

**Proof:** Instead of GMININF, we will consider the corresponding problem GMINEXT, proving its tractability. Since GMININF is the co-problem of GMINEXT, its tractability follows.

The bijunctive affine relations are model sets of affine systems containing equations of the following four types:

$$x + y = 0, \quad x + y = 1, \quad x = 0, \quad x = 1.$$

Equations of type  $x + y = 0$  imply  $x = y$ . Hence we can (1) eliminate the relation represented by this equation, (2) identify the variables  $x$  and  $y$  in the rest of the problem, and (3) check its compatibility with the partial assignment  $m$ .

The equations of the type  $x = 0$  and  $x = 1$  fix a value for a given variable. Hence, we can (1) eliminate the relation represented by this equation, (2) substitute the given value for  $x$  in the rest of the problem, and (3) check its compatibility with the partial assignment  $m$ .

If we have two equations  $x + y = 1$  and  $y + z = 1$  with one overlapping variable  $y$ , then we can replace the second equation by  $x + z = 0$ . The conjunction  $(x + y = 1) \wedge (x + z = 0)$  is equivalent through Gaussian elimination to the original conjunction  $(x + y = 1) \wedge (y + z = 1)$ . We can treat now the equation  $x + z = 0$  as above.

At the end, we get a conjunction of relations built only from the relation  $[x + y = 1]$ , where each variable occurs in the system exactly once. Check if the partial assignment  $m$ , modulo aforementioned variable identifications, is extensible to a satisfying assignment. If YES, the partial assignment  $m$  can be extended to a minimal one, OTHERWISE it cannot be extended.  $\square$

We consider now the case relations generated by clauses with only negative literals, also called negative Horn.

**Proposition 5.4**  $\text{GMININF}(iS_1)$  is decidable in polynomial time.

**Proof:** Once more we consider the problem  $\text{GMINEXT}$  instead of  $\text{GMININF}$ . Recall that the co-clone  $iS_1$  consists of all relations corresponding to negative Horn formulas  $\varphi$ . i.e., relations of the form  $\{0, 1\}^n \setminus \{1 \cdots 1\}$ . Since  $\varphi$  is negative Horn, it is easy to decide whether a partial assignment  $m$  can be extended to a satisfying assignment  $\bar{m}$ . It should also be clear that such a partial assignment can be extended to a minimal assignment if and only if all variables from  $P$ , which are assigned by this partial assignment  $m$ , are assigned to the value 0. This is because in any satisfying assignment  $m'$  with  $m'(x) = 1$  there exists another satisfying assignment of  $\varphi$  which coincides with  $m'$  on all variables except  $x$  and where  $x$  is assigned to 0. Hence,  $\text{GMINEXT}(iS_1)$  and therefore also  $\text{GMININF}(iS_1)$  are both polynomial-time decidable.  $\square$

The present the complexity of  $\text{GMININF}(S)$  for 8 particular co-clones.

**Proposition 5.5**  $\text{GMININF}(S)$  is coNP-complete when  $S$  is one of the following co-clones:  $iS_{11}^2$ ,  $iS_0^2$ ,  $iL$ ,  $iV$ , or  $iE$ .  $\text{GMININF}(S)$  is polynomial-time decidable when  $S$  is one of the following co-clones:  $iS_{12}$ ,  $iD_1$ , or  $iM_2$ .

**Proof:** The least upper bound of  $iS_{11}^2$  and  $iR_2$  is  $iS_{10}^2$ . The co-clone  $iS_{10}^2$  contains all relations generated by Horn formulas with at most 2 literals per clause. Cadoli and Lenzerini proved in [6] that  $\text{GMININF}(iS_{10}^2)$  is coNP-complete, hence by Lemma 5.2 it follows that  $\text{GMININF}(iS_{11}^2)$  is coNP-complete.

The co-clone  $iS_0^2$  contains the relations generated by formulas with clauses of the form  $(x \vee y)$ , i.e., containing two positive literals. Cadoli and Lenzerini proved in [6] that  $\text{GMININF}(iS_0^2)$  is coNP-complete.

The least upper bound of the co-clones  $iL$  and  $iR_2$  is  $iL_2$ , the set of all relations generated by affine systems. Durand and Hermann proved in [11] that  $\text{GMININF}(iL_2)$  is coNP-complete, hence by Lemma 5.2 it follows that  $\text{GMININF}(iL)$  is coNP-complete.

The least upper bound of the co-clones  $iV$  and  $iR_2$  is  $iV_2$ , the set of all relations generated by dual Horn formulas. Since  $iS_0^2 \subseteq iV_2$  holds and  $\text{GMININF}(iS_0^2)$  is coNP-complete, it follows, using Lemma 5.2, that  $\text{GMININF}(iV_2)$  is coNP-complete.

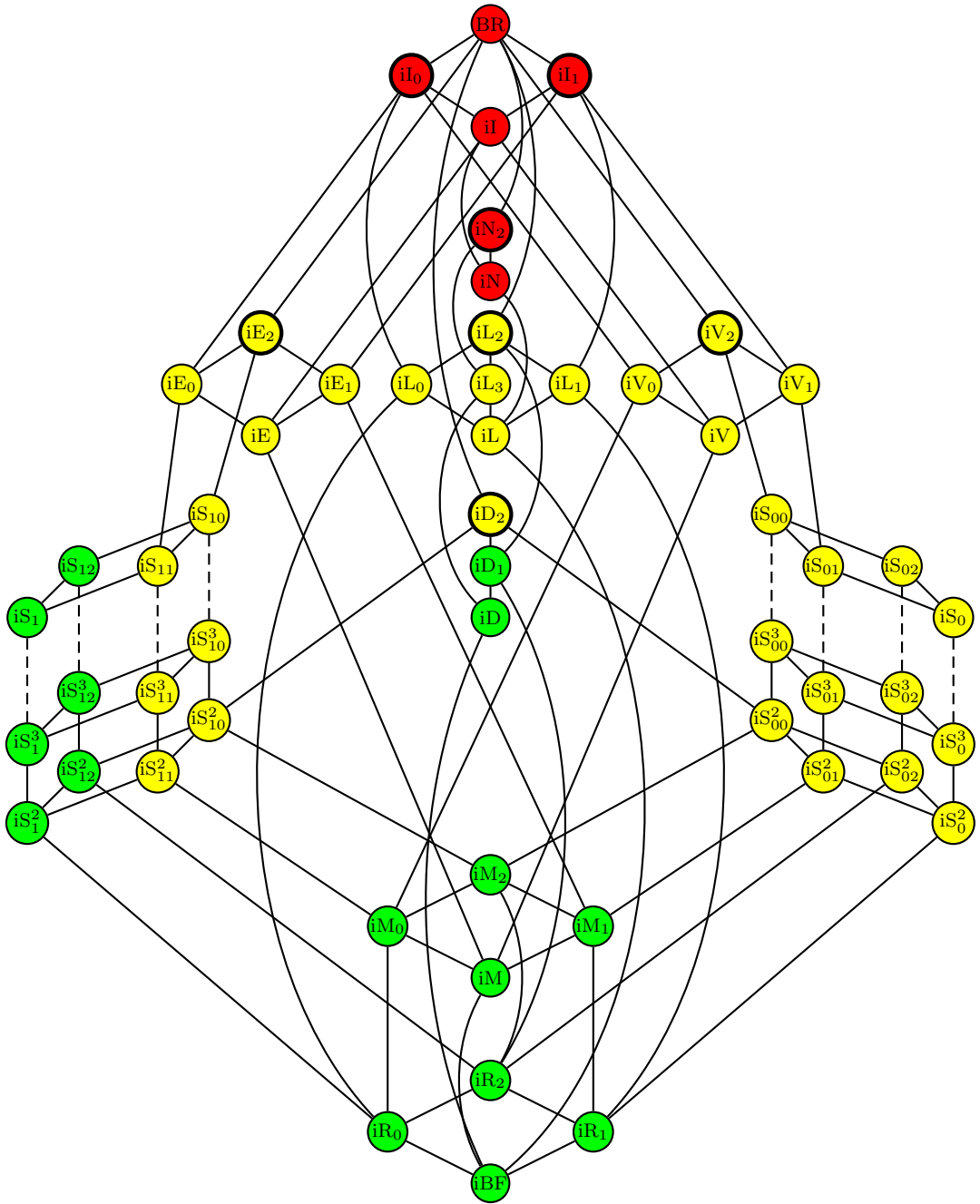
The least upper bound of the co-clones  $iE$  and  $iR_2$  is  $iE_2$ , the set of all relations generated by Horn formulas. Since  $iS_{11}^2 \subseteq iE_2$  holds and  $\text{GMININF}(iS_{11}^2)$  is coNP-complete, using Lemma 5.2 it follows that  $\text{GMININF}(iE)$  is coNP-complete.

We proved in Proposition 5.4 that  $\text{GMININF}(iS_1)$  is in P. The least upper bound of the co-clones  $iS_1$  and  $iR_2$  is  $iS_{12}$ , therefore using Lemma 5.2 it follows that  $\text{GMININF}(iS_{12})$  is in P.

The membership of  $\text{GMININF}(iD_1)$  in P is proved in Proposition 5.3.

The co-clone  $iM_2$  consists of all relations closed under the monotone functions, i.e., the relations generated by formulas which are both Horn and dual Horn. Cadoli and Lenzerini proved in [6] that  $\text{GMININF}(iM_2)$  is in P.  $\square$

In the previous proof we refer to results of Cadoli and Lenzerini in [6] concerning  $\text{GMININF}$ . Of course, they did not consider the problem as we defined it in our paper, but proved equivalent results from which the required result concerning  $\text{GMININF}$  can be obtained by easy syntactic reformulations.



$\Pi_2P$ -complete

coNP-complete

in P

Figure 2: Trichotomy of GMININF

The previous proposition together with the structure of Post's lattice of co-clones and Theorem 5.1 yields the following trichotomy result for the complexity of  $\text{GMININF}(S)$ .

**Theorem 5.6 (Trichotomy of GMININF)** *Let  $S$  be a nonempty finite set of logical relations. If every relation in  $S$  is Horn and dual Horn, bijunctive affine, or negative Horn, then  $\text{GMININF}(S)$  is decidable in polynomial time. Else if every relation in  $S$  is dual Horn, bijunctive, or affine, then  $\text{GMININF}(S)$  is coNP-complete. Otherwise  $\text{GMININF}(S)$  is  $\Pi_2\text{P}$ -complete.*

Figure 2 shows the complexity results for  $\text{GMININF}(S)$  visualized in the lattice of co-clones. The co-clones colored in red correspond to  $\Pi_2\text{P}$ -complete problems, those colored in yellow correspond to coNP-complete problems, and finally those colored in green correspond to polynomial-time decidable problems.

## 6 Minimal Inference With Free Variables But Without Constants

We want to derive in this section a characterization of the complexity for the  $\text{VMININF}$  problem under restrictions on the types of allowed constraints. Theorem 4.2 is valid in this context, therefore we can use the algebraic approach working with clones and co-clones. Since the polynomial-time decidable cases are known and a dichotomy between  $\Pi_2\text{P}$ -complete cases of  $\text{VMININF}$  and those included in coNP (almost) carries over from [18], it is the case that the most involved proofs concern the intermediate level, especially the case of affine relations. Infact, the coNP-hardness proofs for affine relations that we present here are a significant strengthening of Durand's and Hermann's main result from [11]. Moreover, our proofs are much simpler than the rather involved proof in [11].

### 6.1 Complementary or 1-valid Relations

Note first that Kirousis and Kolaitis show in [18] the existence of 1-valid relations  $S$  (i.e., relations from  $iI_1$ ) for which the problem  $\text{MININF}(S)$  is  $\Pi_2\text{P}$ -complete. Using Theorem 4.2 we obtain the following result.

**Proposition 6.1**  $\text{VMININF}(iI_1)$  is  $\Pi_2\text{P}$ -complete.

We continue with complementary relations. Let  $R \subseteq \{0, 1\}^n$  be an  $n$ -ary Boolean relation,  $R^0$  the closure of  $R$  under a 0-prefix, and  $R^\neg$  the closure of  $R$  under negation, i.e.,

$$\begin{aligned} R^0 &= \{(0, a_1, \dots, a_n) \mid (a_1, \dots, a_n) \in R\} \\ R^\neg &= R \cup \{(1 - a_1, \dots, 1 - a_n) \mid (a_1, \dots, a_n) \in R\} \end{aligned}$$

Define  $R^{0\neg} = (R^0)^\neg$  to be the  $(n+1)$ -ary relation produced from  $R$  by a 0-prefix and closure under negation.

**Lemma 6.2** *If  $\text{Pol}(R) = I_2$  then  $\text{Pol}(R^\neg) = N_2$ .*

**Proof:** It is clear that  $R^\neg$  is closed under negation, resulting from the closure construction. We investigate the possibility of  $\text{Pol}(R^\neg)$  to reside in a larger clone than  $N_2$ .

Suppose that  $\text{Pol}(R^\neg) \supseteq N$ , hence  $R^\neg$  is both 0- and 1-valid. Therefore  $R$  must be 0- or 1-valid, what is a contradiction.

Suppose that  $\text{Pol}(R^\neg) \supseteq L_3$ . Consider the existence of three vectors  $a$ ,  $b$ , and  $c$  from  $R^\neg$ . From the pigeonhole principle it follows that either (1)  $a, b \in R$  and  $\bar{c} \in R^\neg \setminus R$ , or (2)  $a \in R$  and  $\bar{b}, \bar{c} \in R^\neg \setminus R$ . If the second choice has been made, then there exist the vectors  $\neg a \in R^\neg \setminus R$  and  $\neg \bar{b}, \neg \bar{c} \in R$ , hence we can restrict ourselves to the first case.

Let  $a = (a_1, \dots, a_n)$ ,  $b = (b_1, \dots, b_n)$ , and  $\bar{c} = (1 - c_1, \dots, 1 - c_n)$  be the considered vectors for some  $c = (c_1, \dots, c_n) \in R$ . The co-clone  $iL_3$  is closed under the operation  $x + y + z + 1 \pmod 2$ , therefore we must have the vector  $d = (d_1, \dots, d_n) \in R^\neg$ , where

$$\begin{aligned} d_i &= a_i + b_i + (1 + c_i) + 1 \pmod 2 \\ &= a_i + b_i + c_i \pmod 2 \end{aligned}$$

for all  $i = 1, \dots, n$ . If the vector  $d$  belongs to  $R$ , then  $R$  must be included in  $iL_2$ , what is a contradiction. If  $d \in R^\neg \setminus R$ , then the complement  $\bar{d} = (1 - d_1, \dots, 1 - d_n)$  must be in  $R$ , i.e.,  $\bar{d}_i = 1 - d_i = a_i + b_i + c_i + 1 \pmod 2$ . But then  $R$  must be included in  $iL_3$ , what is once more a contradiction.  $\square$

**Lemma 6.3** *If  $\text{Pol}(R) = I_2$  then  $\text{Pol}(R^0) = I_2$ .*

**Proof:** Let  $R$  be an  $n$ -ary Boolean relation and  $R^0$  its  $(n+1)$ -ary extension by 0-prefix. Suppose that  $R^0$  is closed under a Boolean  $k$ -ary function  $f$  other than the identity. This means that

$$(f(a_1[0], \dots, a_k[0]), f(a_1[1], \dots, a_k[1]), \dots, f(a_1[n], \dots, a_k[n])) \in R^0$$

for any not necessarily distinct  $k$ -tuple of vectors  $a_1, \dots, a_k \in R^0$ . However, this also implies

$$(f(a_1[1], \dots, a_k[1]), \dots, f(a_1[n], \dots, a_k[n])) \in R$$

what is in contradiction with  $\text{Pol}(R) = I_2$ .  $\square$

**Proposition 6.4**  *$\text{VMININF}(iN_2)$  is  $\Pi_2P$ -complete.*

**Proof:** We know from [12] that there exists a relation  $R$  for which  $\text{MININF}(R)$  is  $\Pi_2P$ -complete, hence  $\text{MINEXT}(R)$  is  $\Sigma_2P$ -complete. Let

$$\varphi(\vec{x}, \vec{y}) = \bigwedge_{i=1}^k R(\vec{x}, \vec{y})$$

be a CNF-formula and  $m = (m_1, \dots, m_n)$  a partial assignment to the variables  $\vec{x}$ . Let

$$\varphi'(z, \vec{x}, \vec{y}) = \bigwedge_{i=1}^k R^{0^\neg}(z, \vec{x}, \vec{y})$$

be a CNF-formula and  $m' = (0, m_1, \dots, m_n)$  a partial assignment to the variables  $(z, \vec{x})$ . It is clear that the vector  $m$  can be extended to a minimal assignment of  $\varphi$  if and only if  $m'$  can be extended to a minimal assignment of  $\varphi'$ . Indeed, if  $\text{Pol}(R) = I_2$  then  $\text{Pol}(R_i^{0^\neg}) = N_2$  for each  $i = 1, \dots, k$ , following Lemmas 6.3 and 6.2. Since the formula  $\varphi'$  is a conjunction of complementary relations, using Theorem 4.2 this proves the result.  $\square$

## 6.2 0-valid Relations

We begin by observing that  $\text{VMININF}$  for 0-valid relations is in  $\text{coNP}$ .

**Proposition 6.5**  $\text{VMININF}(iI_0)$  is in  $\text{coNP}$ .

**Proof:** Any formula  $\varphi(x_1, \dots, x_n)$  built over a set of 0-valid relations has at least one minimal model  $m$  which is  $m(x_i) = 0$  for all  $i$ . Let  $\varphi$  be the theory,  $\psi$  the clause to be minimally inferred and  $P, Z$  the partition of the variables into variables to be minimized and free variables, respectively. If  $\psi$  contains a literal  $\neg x$  such that  $x \in P$ , then obviously  $\psi$  is true in any minimal model of  $\varphi$ . If  $\psi$  consists only of positive literals, then  $\psi$  cannot be minimally inferred from  $\varphi$  since the all 0 minimal model is not a model of  $\psi$ . Otherwise, let  $\varphi_0$  ( $\psi_0$ ) be the formula (clause) resulting from assigning 0 to all the variables in  $P$ . Obviously  $\psi$  can be minimally inferred from  $\varphi$  if and only if  $\psi_0$  can be (classically) inferred from  $\varphi_0$ . The result follows since classical inference is in  $\text{coNP}$ .  $\square$

**Proposition 6.6**  $\text{VMININF}(iN)$  is  $\text{coNP}$ -complete.

**Proof:** We present a polynomial-time reduction from the NP-complete problem  $\text{NAE-3-SAT}$  [27], i.e.,  $\text{SAT}(nae)$  where  $nae = \{0, 1\}^3 \setminus \{000, 111\}$ , to the complement of  $\text{VMININF}(iN)$ . Given an instance  $\rho(X)$  of  $\text{SAT}(nae)$ , let  $(\varphi, \psi, P, Z)$  be the instance of  $\text{VMININF}(iN)$  where  $Z = X \cup \{x_q\}$ ,  $P = \{x_m\}$ , and  $\psi = \neg x_q$  (i.e., the clause to be inferred is the single negative literal  $\neg x_q$ ), where  $x_q, x_m$  are fresh variables not occurring in  $X$ . Let  $R$  be the following 5-ary relation

$$R = \{00000, 11111\} \cup \{0, 1\} \times nae \times \{0, 1\}.$$

which is in  $iN$  since it is 0-valid, 1-valid, and complementive. For each constraint  $nae(x_i, x_j, x_k)$  in  $\rho(X)$  let  $R(x_m, x_i, x_j, x_k, x_q) \in \varphi$ . Now  $\psi$  cannot be minimally inferred from  $\varphi$  if and only if  $\varphi \wedge \neg x_m \wedge x_q$  is satisfiable. This is because  $R$  is 0-valid, and thus,  $x_m$  must have the value 0 in every minimal model of  $\varphi$ . It is easy to realize that  $\varphi \wedge \neg x_m \wedge x_q$  is satisfiable if and only if the original instance  $\rho(X)$  of  $\text{SAT}(nae)$  is satisfiable.  $\square$

The following result follows directly from co-clone inclusions in Post's lattice, Theorem 4.2 and Propositions 6.5 and 6.6.

**Corollary 6.7** If  $iN \subseteq \langle S \rangle \subseteq iI_0$  holds then  $\text{VMININF}(S)$  is  $\text{coNP}$ -complete.

## 6.3 Dual Horn and Bijunctive Relations

For the instances of  $\text{VMININF}$  included in  $\text{coNP}$ , it is more convenient to consider the dual instances  $\text{VMINEXT}$  in  $\text{NP}$ . We consider the problem  $\text{VMINEXT}(iS_0^2)$ . Note that the clone  $S_0^2$  includes the clone  $D_2$ , corresponding to bijunctive relations, as well as the clone  $V_1$  corresponding to 1-valid dual Horn relations. We need first a lemma that exhibits an NP-complete problem for a relation belonging to the co-clone  $iS_0^2$ , which was already proved by Cadoli and Lenzerini in [6, Theorem 5].

**Lemma 6.8** ([6])  $\text{MINEXT}(R)$  is NP-complete for  $R = [x \vee y]$ .

**Proposition 6.9**  $\text{VMININF}(iS_0^2)$  is  $\text{coNP}$ -complete.

**Proof:** We have  $[x \vee y] \in iS_0^2$  since  $[x \vee y] = \{01, 10, 11\}$  is closed under both the implication function and maj. In the same time we have  $[x \vee y] \notin R_1$  since  $R_1 \supseteq E_2$  holds and  $[x \vee y]$  is *not* closed under conjunction, because  $01 \wedge 10 = 00 \notin [x \vee y]$ . The rest of the proof follows from Lemma 6.8, Theorem 4.2, and the correspondence between VMINEXT and VMININF.  $\square$

## 6.4 Affine Relations

We consider the problems  $\text{VMININF}(iL_1)$  as well as  $\text{VMININF}(iL_3)$ . We need first some supporting lemmas to exhibit NP-complete problems built from relations in the corresponding co-clones. The result presented in the supporting lemma is a significant strengthening of the previous result due to Durand and Hermann [11], which states that there exists a set of affine relations  $S$  for which  $\text{MININF}(S)$  is coNP-complete. Moreover, the previous proof is rather involved in comparison to the proofs we present here.

**Lemma 6.10**  $\text{MINEXT}(R)$  is NP-complete for  $R = [x + y + z + w = 1]$ .

**Proof:** The equation  $x + y + z + w = 1$  gives rise to  $x + y = 1$  when we identify the variables  $y, z,$  and  $w$ . The membership in NP is clear and we prove NP-hardness by a reduction from 3SAT.

Let  $\varphi(x_1, \dots, x_n) = c_1 \wedge \dots \wedge c_k$  be a 3SAT formula. Associate the variables  $y_i$  with the clauses  $c_i$  for  $i = 1, \dots, k$ . Construct the system in the following way. Add the equations  $x_i + x'_i = 1$  for  $i = 1, \dots, n$  to the constructed formula  $\varphi'$ , what forces  $x_i$  and  $x'_i$  to take opposite values. For each clause  $c_i = l_i^1 \vee l_i^2 \vee l_i^3$  add the following three equations

$$z_{3i-2} + v_i^1 + y_i + w = 1, \quad z_{3i-1} + v_i^2 + y_i + w = 1, \quad z_{3i} + v_i^3 + y_i + w = 1 \quad (1)$$

to the formula  $\varphi'$ , where

$$v_i^j = \begin{cases} x_p & \text{if } l_i^j = \neg x_p, \\ x'_p & \text{if } l_i^j = x_p. \end{cases}$$

The variable  $v_i^j$  is a placeholder for the negation  $\neg l_i^j$  of the literal  $l_i^j$ . The variable  $w$  is the same all over the system.

**Claim:** The formula  $\varphi$  is satisfiable if and only if the assignment  $s(y_i) = 1$  for all  $i = 1, \dots, k$  of  $\varphi'$  has a minimal extension.

**Let  $\varphi$  be satisfiable.** Then there exists a satisfying assignment  $m$  of the formula  $\varphi$ . Since every clause  $c_i$  is satisfied, for each  $i$  there must be a  $j$  such that  $m(l_i^j) = 1$ . Let  $\bar{m}$  be an extension of  $m$  that satisfies  $\varphi'$ . Following the construction of  $\varphi'$ , there must be a variable  $v_i^j$  such that  $\bar{m}(v_i^j) = 0$ . Then there are three possibilities to get a minimal assignment  $\bar{m}$ :

1. When we set  $\bar{m}(z_{3i+3-j}) = 1$ ,  $\bar{m}(y_i) = 0$ , and  $\bar{m}(w) = 0$ , we will get an assignment which is not interesting for us.
2. When we set  $\bar{m}(z_{3i+3-j}) = 0$ ,  $\bar{m}(y_i) = 0$ , and  $\bar{m}(w) = 1$ , we will also get an assignment which is not interesting for us.

3. When we set  $\bar{m}(z_{3i+3-j}) = 0$ ,  $\bar{m}(y_i) = 1$ , and  $\bar{m}(w) = 0$ , we get an assignment which is an extension of the assignment  $s(y_i) = 1$  for all  $i = 1, \dots, k$ .

These three possible assignments are clearly incomparable and no value can be changed from 1 to 0 to get another satisfying assignment of  $\varphi'$ . Therefore the assignment  $\bar{m}$  from the third case is minimal.

**Let  $\varphi$  be unsatisfiable.** Then for each assignment  $m$  there exists a falsified clause  $c_i$  in  $\varphi$ . This means that we have  $m(l_i^j) = 0$  for each  $j = 1, 2, 3$ . Let  $\bar{m}$  be an extension of  $m$  that satisfies the formula  $\varphi'$ . Following the construction of  $\varphi'$ , we have  $\bar{m}(v_i^j) = 1$  for each  $j$ . Then we can set  $\bar{m}(z_{3i+3-j}) = 0$ ,  $\bar{m}(y_i) = 0$ , and  $\bar{m}(w) = 0$  for all  $j$ . This implies that the assignment  $s(y_i) = 1$  for all  $i$  cannot be extended to a minimal one.  $\square$

**Proposition 6.11**  $\text{VMININF}(iL_3)$  is coNP-complete.

**Proof:** The relation  $[x + y + z + w = 1] = \{0001, 0010, 0100, 1000, 0111, 1011, 1101, 1110\}$  is affine and complementive. It is neither 1-valid nor 0-valid and it is *not* bijunctive since the closure under majority computes  $\text{maj}(0001, 0010, 0100) = 0000 \notin [x + y + z + w = 1]$ . The proof is finished by means of Lemma 6.10, Theorem 4.2, and the correspondence between  $\text{VMINEXT}$  and  $\text{VMININF}$ .  $\square$

**Lemma 6.12**  $\text{MINEXT}(R)$  is NP-complete for the relations  $R = [(x + y + z = 1) \wedge (x + w = 1)]$  and  $R = [(x + y + z = 0) \wedge (x + w = 1)]$ .

**Proof:** The membership in NP is clear, we focus on the NP-hardness proof by means of a reduction from 3SAT. Let  $\varphi(x_1, \dots, x_n) = c_1 \wedge \dots \wedge c_k$  be a 3SAT formula. Associate the variables  $y_i$  with the clauses  $c_i$  for  $i = 1, \dots, k$ . Construct the system in the following way.

For each clause  $c_i = l_i^1 \vee l_i^2 \vee l_i^3$  add the following three formulas

$$v_i^j = \begin{cases} x_p & \text{if } l_i^j = \neg x_p, \\ x'_p & \text{if } l_i^j = x_p. \end{cases} \quad \text{and} \quad u_i^j = \begin{cases} x'_p & \text{if } l_i^j = \neg x_p, \\ x_p & \text{if } l_i^j = x_p. \end{cases}$$

The variable  $v_i^j$  is a placeholder for the negation  $\neg l_i^j$  of the literal  $l_i^j$  and vice versa for  $u_i^j$ .

**Claim:** The formula  $\varphi$  is satisfiable if and only if the assignment  $s(y_i) = 1$  for all  $i = 1, \dots, k$  of  $\varphi'$  has a minimal extension.

**Let  $\varphi$  be satisfiable.** Let  $m$  be a satisfying assignment of  $\varphi$ . Then in each clause  $c_i$  there must be a literal  $l_i^j$  such that  $m(l_i^j) = 1$ . Hence, for each extension  $\bar{m}$  of  $m$  that satisfies  $\varphi'$  we must have  $\bar{m}(v_i^j) = 0$  following the definition of  $v_i^j$ , where the equation  $x + x' = 1$  enforces the variables  $x$  and  $x'$  to take opposite values. The value  $\bar{m}(v_i^j) = 0$  implies two different incomparable assignments for the variables  $y_i$  and  $z_{3i+3-j}$ . The first one is  $\bar{m}(z_{3i+3-j}) = 1$  and  $\bar{m}(y_i) = 0$ , which is not interesting for us. The second is  $\bar{m}(z_{3i+3-j}) = 0$  and  $\bar{m}(y_i) = 1$ , which is the desired minimal assignment. It is clear from the construction of the formula  $\varphi'$  that this assignment is minimal. It is also clear that  $\bar{m}$  is an extension of the assignment  $s(y_i) = 1$  for all  $i = 1, \dots, k$ . Hence, the assignment  $s(y_i) = 1$  for all  $i$  can be extended to a minimal one.

**Let  $\varphi$  be unsatisfiable.** Then for all assignments  $m$  of the variables  $V$  there exists a falsified clause  $c_i$ . This implies that  $m(l_i^j) = 0$  for all  $j = 1, 2, 3$ . Let  $\bar{m}$  be an extension of  $m$  that satisfies the formula  $\varphi'$ . Then the structure of the formula  $\varphi'$  implies that we have  $\bar{m}(v_i^1) = \bar{m}(v_i^2) = \bar{m}(v_i^3) = 1$ . This implies the existence of a minimal assignment with  $y_i = 0$  and  $\bar{m}(z_{3i+3-j}) = 0$  for all  $j$ . Hence, the assignment  $s(y_i) = 1$  for all  $i$  cannot be extended to a minimal one.

Finally for the latter relation, by swapping variables we get  $[(x + y + z = 0) \wedge (x + w = 1)] = [(w + y + z = 1) \wedge (x + w = 1)]$  and the result follows from the previous relation.  $\square$

**Lemma 6.13**  $\text{MINEXT}(R)$  is NP-complete for  $R = [x+y+z = 1]$  and  $R = [(x+y+z = 1) \wedge (w = 1)]$ .

**Proof:** We proceed by dropping the relation  $[x + y = 1]$  from  $S$  in Lemma 6.12. This means that we reformulate the proof using only the relation  $[x + y + z = 1]$ . Write the equation  $x_i'' + x_i + x_i' = 1$  instead of  $x_i + x_i' = 1$  for each  $i$ . Add also the equation  $x_i'' + u_i + w_i = 1$  for each  $i$ , where  $u_i$  and  $w_i$  are new variables.

Define the partial assignment to be  $s(y_i) = 1$  and  $s(u_i) = 1$  for each  $i$ . Let  $m$  be a minimal assignment of the formula  $\varphi'$ . Suppose that there exists an  $i$ , such that  $m(x_i'') = 1$ . Then we have also  $m(w_i) = 1$ . But then we can construct a new assignment  $m'$  with  $m'(u_i) = 0$ ,  $m'(w_i) = 0$ , and  $m'(x_i'') = 1$ . This is a contradiction to the assumption that the partial assignment  $s$  can be extended to a minimal one. Hence, for all  $i$  we must have  $m(x_i'') = 0$ .

The relation  $R'(x, y, z, w) = [(x + y + z = 1) \wedge (w = 1)]$  can be constructed from the relation  $R(x, y, z) = [x + y + z = 1]$  as  $R'(x, y, z, w) = R(x, y, z) \wedge R(w, w, w)$ . Since the value of the variable  $w$  is determined by the trivial equation  $w = 1$ , the complexity of  $\text{MINEXT}(R')$  is equal to that of  $\text{MINEXT}(R)$ .  $\square$

**Proposition 6.14**  $\text{VMININF}(iL_1)$  is coNP-complete.

**Proof:** For the relation  $[x + y + z = 1]$  from Lemma 6.13 we know that  $[x + y + z = 1] \in iL_1$ , since the relation is linear and 1-valid. We also know that  $[x + y + z = 1] \notin iL$ , since  $[x + y + z = 1]$  is not 0-valid. Finally,  $[x + y + z = 1] \notin iR_1$  since the relation  $[x + y + z = 1] = \{001, 010, 100, 111\}$  is not closed under disjunction, i.e.,  $001 \vee 010 = 011 \notin [x + y + z = 1]$ . The proof is finished by means of Lemma 6.13, Theorem 4.2, and the relation between  $\text{VMINEXT}$  and  $\text{VMININF}$ .  $\square$

## 6.5 Polynomial-Time Decidable Cases

The polynomial-time decidable cases for  $\text{GMINEXT}(S)$  propagates to  $\text{VMINEXT}(S)$ . There are three other cases of  $\text{VMINEXT}(S)$  that are also decidable in polynomial time.

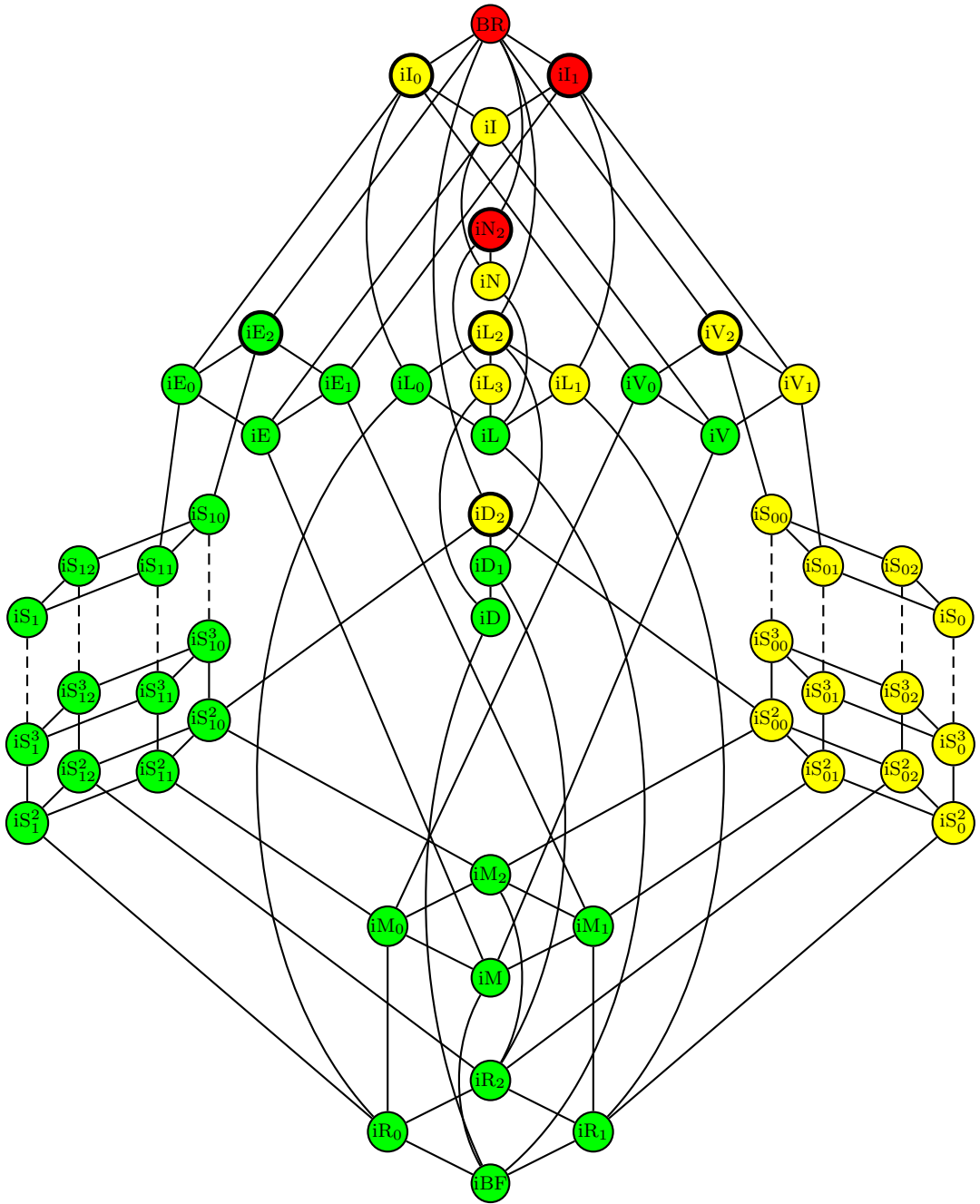
Tractability of  $\text{VMINEXT}(S)$  when for sets of Horn relations  $S$  was already showed by Cadoli and Lenzerini in [6]. This implies the following result.

**Proposition 6.15** ([6])  $\text{VMININF}(iE_2)$  is decidable in polynomial time.

**Proposition 6.16**  $\text{VMININF}(iL_0)$  and  $\text{VMININF}(iV_0)$  are in P.

**Proof:** We only give the proof for  $iL_0$  since the proof for  $iV_0$  uses the exact same argument.

Given an instance  $(\varphi, \psi, P, Z)$  of  $\text{VMININF}(iL_0)$ , we know that the all-zero assignment  $0 \cdots 0$  is a minimal model of  $\varphi$  since  $iL_0$  is 0-valid. If  $\psi$  contains a literal  $\neg x$  such that  $x \in P$ , then



$\Pi_2P$ -complete

coNP-complete

in P

Figure 3: Trichotomy of VMININF

obviously  $\psi$  is true in any minimal model of  $\varphi$ . If  $\psi$  consists only of positive literals, then  $\psi$  cannot be minimally inferred from  $\varphi$  since the all-zero minimal model is not a model of  $\psi$ .

Otherwise, we can reduce the problem by assigning 0 to all the variables in  $P$ ; resulting in the equivalent instance  $(\varphi_0, \psi_0, \emptyset, Z)$ . The clause  $\psi$  can be minimally inferred from  $\varphi$  if and only if  $\psi_0$  can be (classically) inferred from  $\varphi_0$ . Note that  $\varphi_0$  is still a formula built from relations in  $iL_0$  (i.e., affine and 0-valid), and since inference over affine constraints is polynomial-time decidable it follows that  $\text{VMININF}(iL_0)$  is in P.  $\square$

## 6.6 Main Result

We put all results of this section together and we get the following trichotomy result for minimal inference with free variables and without constants.

**Theorem 6.17 (Trichotomy of VMININF)** *Let  $S$  be a nonempty finite set of logical relations. If every relation in  $S$  is Horn, or affine and 0-valid, or dual Horn and 0-valid, or bijunctive affine, then  $\text{VMININF}(S)$  is decidable in polynomial time. Else if every relation in  $S$  is 0-valid, or dual Horn, or bijunctive, or affine, then  $\text{VMININF}(S)$  is coNP-complete. Otherwise  $\text{VMININF}(S)$  is  $\Pi_2P$ -complete.*

Figure 3 shows the complexity results for  $\text{VMININF}(S)$  visualized in the lattice of co-clones. The co-clones colored in red correspond to  $\Pi_2P$ -complete problems, those colored in yellow correspond to coNP-complete problems, and finally those colored in green correspond to polynomial-time decidable problems.

## 7 Expressivity Techniques

The problems  $\text{CMININF}$  and  $\text{MININF}$  are not compatible with the co-clone closure  $\langle S \rangle$  of a set of relations  $S$  any more, since the application of existential quantification to a constraint can change the order of the models in the corresponding relation, but only with the weak co-clone closure  $\langle S \rangle_{\#}$ . For this reasons we need to review in this section techniques compatible with the weak co-clone closure  $\langle S \rangle_{\#}$ . The first is due to Schaefer [27], for showing that certain relations belong to  $\langle S \rangle_{\#}$  based on information about the polymorphisms of  $S$ . The second one, namely the 1-restriction, is due to Kirousis and Kolaitis [18] and it was already mentioned in the preliminaries. We begin by stating some basic but useful facts about the set  $\langle S \rangle_{\#}$ .

**Lemma 7.1** *If  $S$  is not closed under the functions  $f_1, \dots, f_k$  then there exists a single relation  $R \in \langle S \rangle_{\#}$  which is not closed under the functions  $f_1, \dots, f_k$ .*

**Proof:** Since  $S$  is not closed under the functions  $f_1, \dots, f_k$ , there exist relations  $R_1, \dots, R_k$  in  $S$  such that  $R_i$  is not closed under  $f_i$ . Let  $R$  be the Cartesian product of the relations  $R_1, \dots, R_k$ . Obviously  $R$  can be expressed as the conjunction  $R_1(x_{11}, \dots, x_{1m}) \wedge \dots \wedge R_k(x_{k1}, \dots, x_{kn})$ , where  $x_{11}, \dots, x_{kn}$  are all distinct variables. Hence  $R \in \langle S \rangle_{\#}$  holds and  $R$  is not closed under the functions  $f_1, \dots, f_k$ .  $\square$

**Lemma 7.2** *If a relation  $R$  is not closed under the functions  $f_1, \dots, f_k$  of arities  $a_1, \dots, a_k$  where  $a$  denotes the maximum arity of the functions, then there is a relation  $R' \in \langle R \rangle_{\#}$  containing tuples*

$t_1, \dots, t_a$ , such that for any function  $f_i$  and any  $a_i$  tuples from  $t_1, \dots, t_a$ , the tuple resulting from applying  $f_i$  to these  $a_i$  tuples (in any order), is not in  $R'$ .

**Proof:** We prove the result by induction on the number of functions  $k$ . In the base case we have one function  $f_1$ , and we let  $R'$  be  $R^{a_1!}$ . Since we know that  $R$  is not closed under  $f_1$  there exist  $a_1$  tuples  $t_1, \dots, t_{a_1}$  such that  $f_1(t_1, \dots, t_{a_1}) \notin R$ . Let  $S_{a_1}$  denote the group of permutations of  $1, \dots, a_1$ , and denote the  $a_1!$  permutations by  $\pi_1, \dots, \pi_{a_1!}$ . Now construct the  $a_1$  tuples by, for each  $1 \leq i \leq a_1$ , constructing the tuple  $t'_i = t_{\pi_1(i)} \times \dots \times t_{\pi_{a_1!}(i)}$ . Clearly,  $f_1$  applied to the tuples  $t'_1, \dots, t'_{a_1}$  in any order results in a tuple which is not in  $R'$ .

Now, assuming the result holds for  $k - 1$  functions  $f_1, \dots, f_{k-1}$  we show that it must hold also for  $k$  functions  $f_1, \dots, f_k$ . Assume without loss of generality that  $f_j$ , for some  $j < k$ , is a function of maximum arity among  $f_1, \dots, f_k$ . By assumption we know that there is a relation  $R_1 \in \langle R \rangle_{\#}$  and  $a_j = a$  tuples  $t'_1, \dots, t'_{a_j}$  in  $R_1$  such that for any function  $f_i$ , for some  $i < k$ , the function  $f_i$  applied to any  $a_i$  tuples from  $t'_1, \dots, t'_{a_j}$  (in any order) results in a tuple which is not in  $R_1$ . We also know that there is a relation  $R_2 \in \langle R \rangle_{\#}$  and  $a_k$  tuples such that  $f_k$  applied to these tuples in any order results in a tuple which is not in  $R_2$ . By assumption  $a_j \geq a_k$ . By taking Cartesian products of  $R_2$  we get a relation  $R'_2$  containing at least  $a_j$  tuples  $s'_1, \dots, s'_{a_j}$  such that  $f_k$  applied to any  $a_k$  of these  $a_j$  tuples (in any order), results in a tuple that is not in  $R'_2$ . Now take the Cartesian product  $R_1 \times R'_2$  and consider the tuples  $t'_1 \cdot s'_1, \dots, t'_{a_j} \cdot s'_{a_j}$  (where  $\cdot$  denotes tuple concatenation). By the reasoning above we have that  $f_i \in \{f_1, \dots, f_k\}$  applied (in any order) to any  $a_i$  tuples from  $t'_1 \cdot s'_1, \dots, t'_{a_j} \cdot s'_{a_j}$  results in a tuple which is not in  $R_1 \times R'_2$ , and the result follows.  $\square$

The notation  $R[x/V]$  means that in the relation  $R$  the variables  $V$  are replaced with the new variable  $x$ . As a shorthand, we also write  $R[V]$  instead of  $R[x/V]$  when the new variable  $x$  is implicitly clear or when  $V$  represents both the set of variables to be replaced and the fresh variable. This means that  $V$  is a set of variables, but also the identifier of a new variable, by which all variables in  $V$  are replaced in  $R[V]$ . Recall that  $[R(x_1, \dots, x_k)]$  denotes the relation generated by the constraint  $R(x_1, \dots, x_k)$ .

The following is the typical situation we are faced with when proving our hardness results. We have a set of relations  $S$  for which we want to prove a hardness result. We also know a hardness result for a relation  $R$ . The goal is to show that  $R \in \langle S \rangle_{\#}$ . For this purpose we have some information on  $S$ , namely we know that  $S$  is closed under the operations  $f_1, \dots, f_j$ , but it is *not* closed under the operations  $g_1, \dots, g_k$ . The first step is to use Lemmas 7.1 and 7.2 to produce a single relation  $R' \in \langle S \rangle_{\#}$  that has the properties stated in Lemma 7.2 and is not closed under any of the operations  $g_1, \dots, g_k$ .

The next (crucial) step is to prove that  $R$  can be implemented by  $R'$  using variable identification. We first show that  $R'$  must contain a number of vectors  $a_1, \dots, a_p$ , usually by the argument that  $R'$  would otherwise be closed under one of the functions  $g_1, \dots, g_k$ . For all Boolean vectors  $m$  of length  $p$  we construct the variable sets  $V_m = \{x \mid a_1(x) = m[1], \dots, a_p(x) = m[p]\}$ . We then construct the constraint  $Q(V_0, \dots, V_{2^p-1}) = R'[V_i \mid i = 0, \dots, 2^p - 1]$ . For instance, when we know the existence of the vectors  $a, b, c \in R'$ , we create the variable sets  $V_{ijk} = \{x \mid a(x) = i, b(x) = j, c(x) = k\}$  and construct the constraint  $Q(V_0, \dots, V_7) = R'[V_{000}, \dots, V_{111}]$ .

Finally, we must show that the constructed constraint  $R'[V_i \mid i = 0, \dots, 2^p - 1]$ , possibly after additional variable identification, really generates the relation  $R$ . For more information on this technique, see the monograph [8] or Schaefer's original exposition [27]. Note again that our implementations differ from Schaefer's since we are not allowed to use existential quantification to

eliminate variables.

## 8 Minimal Inference Without Free Variables But With Constants

We want to derive in this section a complete complexity characterization of the  $\text{CMININF}$  problem. Theorem 4.2 is not valid any more in this context since there are for example sets of relations  $S$  such that  $\text{CMININF}(S)$  is in  $\text{P}$  but  $\text{CMININF}(\langle S \rangle)$  is  $\text{coNP}$ -complete.

The complexity classification for the  $\text{CMININF}$  problem will be carried out in two phases. First we deal with sets of relations  $S$  that are neither Horn, dual Horn, bijective, nor affine. We call these types of sets  $S$  for non-Schaefer. Then we move on and classify the complexity of  $\text{CMININF}(S)$  for sets of relations  $S$  that are Schaefer (i.e.,  $S$  which are Horn, dual Horn, bijective, or affine).

In the first phase we will use an approach already used by Kirousis and Kolaitis in [18], where we use 1-valid 0-sections, a well-known concept from coding theory and already used for circumscription [17] and minimal inference [18]. The following result is the main theorem in [18].

**Proposition 8.1** ([18]) *Let  $S$  be a finite set of Boolean relations and  $S^*$  the corresponding 1-valid restriction. Then there exists a polynomial many-one reduction from  $\text{CMININF}(S^*)$  to  $\text{CMININF}(S)$ .*

**Proposition 8.2** *Let  $S$  be non-Schaefer. If  $S^*$  is Schaefer then  $\text{CMININF}(S)$  is  $\text{coNP}$ -complete, otherwise  $\text{CMININF}(S)$  is  $\Pi_2\text{P}$ -complete.*

**Proof:** Kirousis and Kolaitis proved in [18] that  $\text{MININF}(S^{\text{id}})$  is  $\Pi_2\text{P}$ -complete when  $S^*$  is not Schaefer. We give a polynomial-time reduction from  $\text{MININF}(S^{\text{id}})$  to  $\text{CMININF}(S)$  proving that  $\text{CMININF}(S)$  is  $\Pi_2\text{P}$ -complete when  $S^*$  is not Schaefer. Given an instance  $\varphi \models_{\text{min}} \psi$  of  $\text{MININF}(S^{\text{id}})$  consisting of a conjunction of constraints  $\varphi$  over  $S^{\text{id}}$  and a clause to be inferred  $\psi$ . For all constraints of the form  $x = 0$  ( $x = 1$ ) we add the literal  $x$  ( $\neg x$ ) to  $\psi$ , remove the constraint  $x = 0$  ( $x = 1$ ), and let  $x$  be fixed (i.e., add  $x$  to  $Q$ ). The idea behind the reduction is as follows. If  $x = 0$  is a constraint in  $\varphi$ , we remove it and modify  $\psi$  to make sure that every minimal model  $m_*$  of  $\varphi \setminus \{x = 0\}$  such that  $m_*(x) = 1$ , is a model of  $\psi$ , and in the case where  $m_*(x) = 0$  we make sure that  $m_*$  is a model of the modified  $\psi$  if and only if  $m_*$  was a model of the original  $\psi$ . The case where  $x = 1$  is a constraint in  $\varphi$  is handled in the same way. It should be clear that the resulting instance is equivalent to the original instance  $\varphi \models_{\text{min}} \psi$ .

Kirousis and Kolaitis proved in [17] that the minimal model checking problem for circumscription with constants but no free variables is in  $\text{P}$  when  $S^*$  is Schaefer, and hence by well-known arguments  $\text{CMININF}(S)$  is in  $\text{coNP}$  when  $S^*$  is Schaefer. It remains to prove that  $\text{CMININF}(S)$  is  $\text{coNP}$ -hard when  $S$  is non-Schaefer. We give a reduction from  $\text{UNSAT}(S^{\text{id}})$ , the complement of the satisfiability problem, to  $\text{CMININF}(S)$ . Since  $\text{UNSAT}(S^{\text{id}})$  is  $\text{coNP}$ -complete when  $S$  is non-Schaefer [27], our result will follow. Given a conjunction of constraints  $\varphi$  over  $S^{\text{id}}$ , let  $\varphi'$  be  $\varphi \wedge (x = 1)$  where  $x$  is a fresh variable. Then  $\neg x$  can be inferred from  $\varphi'$  under propositional circumscription if and only if  $\varphi$  is not satisfiable. This gives a reduction from  $\text{UNSAT}(S^{\text{id}})$  to  $\text{MININF}(S^{\text{id}})$ . Hence when combined with the reduction from  $\text{MININF}(S^{\text{id}})$  to  $\text{CMININF}(S)$ , we get the desired result that  $\text{CMININF}(S)$  is in  $\text{coNP}$ -hard when  $S$  is non-Schaefer.  $\square$

The following result appears in [6] in the form that ECWA-reasoning with constants for formulas that are both Horn and bijective (HK in their notation) is  $\text{coNP}$ -complete.

**Proposition 8.3** ([6])  *$\text{CMININF}(S)$  is  $\text{coNP}$ -complete for  $S = \{[\neg x \vee \neg y], [\neg x \vee y]\}$ .*

## 8.1 A New Tractable Class

In this section we present a new tractable class for  $\text{CMININF}$ .

**Proposition 8.4** *If a set of relations  $S$  is Schaefer and incomparable then  $\text{CMININF}(S)$  is in P.*

**Proof:** Let  $\varphi \models_{\min} \psi$  be an instance of  $\text{CMININF}$  with  $S$  incomparable. It can be shown that  $[\varphi]$  is an incomparable relation. This follows from the fact that incomparable relations are preserved under conjunction and variable identification. Given two formulas  $\varphi_1$  and  $\varphi_2$ , each having only incomparable models, their conjunction  $\varphi_1 \wedge \varphi_2$  have only incomparable models since a model of  $\varphi_1$  can never be extended to two comparable models of  $\varphi_1 \wedge \varphi_2$  (because  $\varphi_2$  has only incomparable models). Variable identification only reduces the set of models of a formula. If the original set of models were all incomparable, then of course the reduced set of models will also be incomparable. Also note that adding some variables to  $Q$  (i.e., letting them be fixed) can never make two incomparable models of a formula comparable.

Since  $\varphi$  is expressed by conjunction and variable identification over an incomparable set of relations  $S$  it follows that all models of  $\varphi$  are incomparable and hence also minimal. Thus,  $\varphi \models_{\min} \psi$  if and only if  $\varphi \models \psi$ , where the latter can be checked in polynomial time since  $S$  is Schaefer.  $\square$

The following example shows the impact of incomparable relations on the tractability of  $\text{CMININF}$  compared to  $\text{VMININF}$  upon the same relation.

**Example 8.5** Let  $R = \{0110, 1001, 1100\}$  then  $R$  is a bijunctive relation but not other Schaefer and hence  $\text{VMININF}(R)$  is coNP-complete. On the other hand  $\text{CMININF}(R)$  is in P since  $R$  is incomparable. Let  $R' = \{00111, 01010, 10001, 11100\}$  then  $R'$  is affine but not other Schaefer and hence  $\text{VMININF}(R')$  is coNP-complete. But again  $\text{CMININF}(R')$  is in P since  $R'$  is incomparable.

**Proposition 8.6**  $\text{CMININF}(N)$  for  $N = \{[\neg x \vee \neg y], [x \not\equiv y]\}$  is in P.

**Proof:** Let  $\varphi \models_{\min} \psi$  be an instance of  $\text{CMININF}(N)$  for  $N = \{[\neg x \vee \neg y], [x \not\equiv y]\}$ . Check first that  $\varphi$  is satisfiable, what can be done in polynomial time since  $S$  is bijunctive. If  $\varphi$  is *not* satisfiable, then  $\varphi \models_{\min} \psi$  is trivially satisfied.

If the formula  $\varphi$  is only produced from the relation  $[\neg x \vee \neg y]$  then it is 0-valid and therefore  $\varphi \models_{\min} \psi$  is in P.

The formula  $\varphi$  is a conjunction of binary constraints of the type  $(\neg x \vee \neg y)$  and  $(x \not\equiv y)$ . Let us consider the structure of  $\varphi$ . If a variable  $v$  occurs only in negative constraints, i.e., only in constraints of the form  $(\neg x \vee \neg y)$ , then  $v$  has the value 0 in all minimal models of  $\varphi$ . Hence, we can drop all clauses  $(\neg v \vee \neg y)$  from  $\varphi$  and remove the variable  $v$  from  $\psi$ . Repeat this replacement until all variables occur in a  $\not\equiv$  constraint.

Let  $\varphi' \models_{\min} \psi'$  be the resulting instance. Assume with the aim of reaching a contradiction that there exist two comparable models  $m_0 < m_1$  of  $\varphi'$ . Then there exists a variable  $x$ , such that  $m_0(x) = 0$  and  $m_1(x) = 1$ . Let  $y$  be a variable occurring together with  $x$  in a  $(x \not\equiv y)$  constraint. Then we have  $m_0(y) = 1$  and  $m_1(y) = 0$ , which implies that  $m_0 \not< m_1$ , constituting a contradiction. Hence, all models of  $\varphi'$  are incomparable. Therefore  $\varphi' \models_{\min} \psi'$  holds if and only if  $\varphi' \models \psi'$  which is decidable in polynomial time since  $\varphi'$  is bijunctive.  $\square$

Compare the result of Proposition 8.6 with the fact that  $\text{VMININF}(N)$  for  $N = \{[\neg x \vee \neg y], [x \not\equiv y]\}$  is coNP-complete.

## 8.2 Dual Horn Relations

We apply Schaefer's technique to obtain the following representation lemma.

**Lemma 8.7** *Let  $R$  be a Boolean relation which is dual Horn, but not Horn. Then we can construct the relation  $[x \vee y]$  from  $R$  by variable identification and constant substitution.*

**Proof:** If  $R$  is dual Horn, but not Horn, then there exists two vectors  $a, b \in R$ , such that  $a \wedge b \notin R$  and  $a \vee b \in R$ . Construct the sets of variables  $V_{ij} = \{x \mid a(x) = i, b(x) = j\}$ . Construct the constraint  $Q(x, y) = R[0/V_{00}, x/V_{01}, y/V_{10}, 1/V_{11}]$ . It is clear that the constraint  $Q(x, y)$  generates the relation  $\{01, 10, 11\} = [x \vee y]$ .  $\square$

**Proposition 8.8**  $\text{cMININF}(S)$  for all sets  $S$  of relations, which are dual Horn but not Horn, is coNP-complete.

**Proof:** From each set of relation  $S$ , which are dual Horn but not Horn, we can produce the relation  $[x \vee y]$  according to Lemma 8.7. Cadoli and Lenzerini [6, Theorem 5] proved that  $\text{MININF}([x \vee y])$  is coNP-complete. Hence, the result follows.  $\square$

## 8.3 Horn But Not Negative Horn Relations

We start with some representation lemmas proved by Schaefer's technique.

**Lemma 8.9** *Let  $R$  be a Boolean relation which is Horn, but not dual Horn. Then we can construct the relation  $[\neg x \vee \neg y]$  from  $R$  by variable identification and constant substitution.*

**Proof:** If  $R$  is Horn, but not dual Horn, then there exist two vectors  $a, b \in R$ , such that  $a \vee b \notin R$  and  $a \wedge b \in R$ . Construct the variable sets  $V_{ij} = \{x \mid a(x) = i, b(x) = j\}$ . Construct the constraint  $Q(x, y) = R[0/V_{00}, x/V_{01}, y/V_{10}, 1/V_{11}]$ . It is clear that the constraint  $Q(x, y)$  generates the relation  $\{00, 01, 10\} = [\neg x \vee \neg y]$ .  $\square$

**Lemma 8.10** *Let  $R$  be a Boolean relation which is Horn, but not negative Horn. Then we can construct the relation  $[\neg x \vee y]$  from  $R$  by variable identification and constant substitution.*

**Proof:** If  $R$  is Horn but not negative Horn, then there exist three vectors  $a, b, c \in R$ , such that  $a \wedge b \in R$ ,  $a \wedge c \in R$ ,  $b \wedge c \in R$ , and  $a \wedge b \wedge c \in R$ , but  $a \wedge (b \vee \neg c) \notin R$ , what also implies  $b \vee \neg c \notin R$ . This is because  $x \wedge (y \vee \neg z)$  is a basis for the co-clone  $iS_{12}$  of negative Horn relations, we must have  $a \wedge (b \vee \neg c) \notin R$ . Since  $a \in R$  holds and  $R$  is Horn (i.e., it is closed under conjunction), we must have  $(b \vee \neg c) \notin R$ , otherwise we would have  $a \wedge (b \vee \neg c) \in R$ . Construct the variable sets  $V_{ijk} = \{x \mid a(x) = i, b(x) = j, c(x) = k\}$ . Construct the constraint

$$Q(x, y) = R[0/V_{000}, 0/V_{001}, 0/V_{010}, 0/V_{011}, x/V_{100}, y/V_{101}, x/V_{110}, 1/V_{111}].$$

It can be easily verified that the constraint  $Q(x, y)$  generates the relation  $\{00, 01, 11\} = [\neg x \vee y]$ .  $\square$

**Proposition 8.11**  $\text{cMININF}(S)$  for all sets  $S$  which are Horn, but neither dual Horn, nor negative Horn, are coNP-complete.

**Proof:** Since  $S$  is Horn, but not dual Horn, there exists a relation  $R_1 \in \langle S \rangle_{\#}$  with the same properties according to Lemma 7.1. From  $R_1$  we can produce the relation  $[\neg x \vee \neg y]$  according Lemma 8.9. Since  $S$  is Horn but not negative Horn, there exists a relation  $R_2 \in \langle S \rangle_{\#}$  with this property. From  $R_2$  we can produce the relation  $[\neg x \vee y]$  according to Lemma 8.10. Hence, we can produce both relations  $[\neg x \vee \neg y]$  and  $[\neg x \vee y]$  from  $S$ . Cadoli and Lenzerini proved in [6] (case HK in their notation) that  $\text{CMININF}(S)$  for  $S = \{[\neg x \vee \neg y], [\neg x \vee y]\}$  is coNP-complete.  $\square$

## 8.4 Affine Relations

We need another representation lemma proved by the usual Schaefer's technique.

**Lemma 8.12** *If  $S$  neither Horn nor dual Horn then we can construct the relation  $[x \neq y]$  from  $S$  by conjunction and variable identification.*

**Proof:** Since  $S$  is neither Horn nor dual Horn, there exists a relation  $R \in \langle S \rangle_{\#}$  with the same properties according to Lemma 7.1. Moreover according to Lemma 7.2,  $R$  contains two tuples  $a, b \in R$  such that  $a \wedge b \notin R$  and  $a \vee b \notin R$ . Construct the variable sets  $V_{ij} = \{x \mid a(x) = i, b(x) = j\}$ . Construct the constraint  $Q(x, y) = R[0/V_{00}, x/V_{01}, y/V_{10}, 1/V_{11}]$ . It is clear that the constraint  $Q(x, y)$  generates the relation  $\{01, 10\} = [x \neq y]$ .  $\square$

We need to investigate a special property of incomparable relations.

**Lemma 8.13** *Given an irredundant affine relation  $R$  which is not incomparable, then any two comparable tuples  $a < b$  in  $R$  must differ in at least two positions. Moreover, there must be a third tuple  $c$  which is not constant on the positions where  $a$  and  $b$  differ.*

**Proof:** We claim that the existence two comparable tuples  $a < b$  differing in just one position implies the relation  $R$  to be redundant. Denote by  $i$  the coordinate on which  $a$  and  $b$  differ. For any tuple  $c$  in  $R$  we construct the tuple  $c' = a + b + c$ , which is identical to  $c$  except that  $c'[i] = \neg c[i]$ . Since  $R$  is affine, the tuple  $c'$  must be in the relation. Hence,  $R$  is redundant since it is of the form  $Q \times \{0, 1\}$ . Thus in any irredundant relation any two comparable tuples must differ in at least two positions.

Now let  $a$  and  $b$  differ on at least two positions, say  $i$  and  $j$ . If all tuples  $c \in R$  are constant on the positions  $i$  and  $j$  where  $a$  and  $b$  differ, i.e.,  $c[i] = c[j]$  for all  $c \in R$ , then  $R$  is again redundant because in particular the columns  $i$  and  $j$  are identical.  $\square$

**Proposition 8.14**  *$\text{CMININF}(S)$  is coNP-hard for each set of relations  $S$ , which is affine but neither incomparable nor other Schaefer.*

**Proof:** By taking Cartesian products of relations in  $S$ , there exists a single relation  $R$  in  $\langle S \rangle_{\#}$  which is affine, but neither incomparable, nor bijunctive, nor Horn, nor dual Horn, according to Lemma 7.1.. Since  $R$  is not incomparable, there exist two tuples  $a$  and  $b$  satisfying the condition  $a < b$ . Without loss of generality we can assume that  $a$  and  $b$  are closest possible, i.e., there is no tuple  $t$  where  $a < t < b$  holds. By Lemma 8.13 we know that  $a$  and  $b$  differ in at least two positions, and that there is a third tuple  $c$  not constant on the coordinates where  $a$  and  $b$  differ.

Since  $R$  is neither Horn nor dual Horn, we can assume by taking a Cartesian product of  $R$  with itself, that  $c$  satisfies the conditions  $a \wedge c \notin R$ ,  $b \wedge c \notin R$ ,  $a \vee c \notin R$ , and  $b \vee c \notin R$ . Form the

Schaefer-style variable identification on  $R$  based on the tuples  $a, b, c$ . Note that since  $a < b$  holds, the variables  $V_{100}$  and  $V_{101}$  will not appear. Thus we get the relation  $R[V_{000}, V_{001}, V_{010}, V_{011}, V_{110}, V_{111}]$ . It can be checked that the variables  $V_{001}, V_{010}, V_{011}, V_{110}$  must all appear. Observe that if  $V_{001}$  does not appear, then  $b \vee c = b \in R$  which is a contradiction. Similarly, if  $V_{110}$  does not appear, then  $a \wedge c = a \in R$  which is a contradiction. Moreover,  $V_{010}$  and  $V_{011}$  must appear since  $c$  is not constant on the coordinates where  $a$  and  $b$  differ.

Since  $R$  is affine, it also contains the tuple  $d = a + b + c = 011001$ . Moreover,  $R$  does not contain the tuple  $t = \text{maj}(a, b, c) = 000111$  since this tuple satisfies the condition  $a < t < b$ . We add the constraint  $(V_{110} \neq V_{001}) \wedge (V_{000} = 0) \wedge (V_{111} = 1)$  to  $R[V_{000}, V_{001}, V_{010}, V_{011}, V_{110}, V_{111}]$ . The resulting constraint contains the tuples 000011, 001111, 010101, 011001, but it does not contain the tuple 000111. There are only three undetermined variables so there can be at most 8 tuples satisfying the affine constraint. Moreover, since the constraint is affine the number of tuples satisfying the constraint is a power of 2. Therefore, since 000111 is not satisfying the constraint, we have that

$$\begin{aligned} P(V_{000}, V_{001}, V_{010}, V_{011}, V_{110}, V_{111}) \\ &= R[V_{001}, V_{010}, V_{011}, V_{110}] \wedge (V_{110} \neq V_{001}) \wedge (V_{000} = 0) \wedge (V_{111} = 1) \\ &= (V_{001} + V_{010} + V_{011} = 0) \wedge (V_{110} + V_{001} = 1) \wedge (V_{000} = 0) \wedge (V_{111} = 1). \end{aligned}$$

The problem  $\text{CMININF}(P)$  for the relation  $P$ , generated by  $P(V_{000}, V_{001}, V_{010}, V_{011}, V_{110}, V_{111})$ , is coNP-hard by Lemma 6.12.  $\square$

## 8.5 Bijunctive Relations

**Lemma 8.15**  $\text{MINEXT}(R)$  for  $R = [(x \vee y) \wedge (x \neq z)]$  is NP-complete.

**Proof:** The membership in NP is clear, we focus on the NP-hardness proof by means of a reduction from 3SAT. Let  $\varphi(x_1, \dots, x_n) = c_1 \wedge \dots \wedge c_k$  be a 3SAT formula. Associate the variable  $y_i$  with the clause  $c_i$  for  $i = 1, \dots, k$ . For each clause  $c_i = l_i^1 \vee l_i^2 \vee l_i^3$  we add the following three formulas

$$(y_i \vee v_i^1) \wedge (v_i^1 \neq \bar{v}_i^1), \quad (y_i \vee v_i^2) \wedge (v_i^2 \neq \bar{v}_i^2), \quad \text{and} \quad (y_i \vee v_i^3) \wedge (v_i^3 \neq \bar{v}_i^3),$$

to  $\varphi'$ , where

$$v_i^j = \begin{cases} x & \text{if } l_i^j = \neg x, \\ x' & \text{if } l_i^j = x. \end{cases} \quad \bar{v}_i^j = \begin{cases} x' & \text{if } l_i^j = \neg x, \\ x & \text{if } l_i^j = x. \end{cases}$$

**Claim:** The partial assignment  $s(y_i) = 1$  for each  $i = 1, \dots, k$  can be extended to a minimal assignment of  $\varphi'$  if and only if  $\varphi$  is satisfiable.

Let  $\varphi$  be satisfiable. Every clause  $c_i$  evaluates to 1. For each clause  $c_i$  there exists a  $j$ , such that  $l_i^j = 1$ . Then  $v_i^j = 0$  which implies  $y_i = 1$ . Moreover, since  $v_i^j \neq \bar{v}_i^j$  holds, every satisfying assignment to  $\varphi'$  is incomparable, hence minimal.

Let  $\varphi$  be unsatisfiable. Then there exists a falsified clause  $c_i$ , i.e.,  $l_i^1 = l_i^2 = l_i^3 = 0$ , which implies  $v_i^1 = v_i^2 = v_i^3 = 1$ . Therefore there exists a satisfying assignment  $m$  of  $\varphi'$  with  $m(y_i) = 0$ . hence  $s(y_i) = 1$  for  $i = 1, \dots, k$  cannot be extended to a minimal solution.  $\square$

We need the following result from [15] stating that it is sufficient to consider binary relations when we consider bijunctive constraint languages.

**Proposition 8.16** ([15]) *Given a  $n$ -ary bijunctive constraint  $R(x_1, \dots, x_n)$  then it is equivalent to  $\bigwedge_{1 \leq i \leq j} R_{ij}(x_i, x_j)$  where  $R_{ij}$  is the projection of the relation  $R$  to the coordinates  $i$  and  $j$ .*

Proposition 8.18 in the sequel is introduced already in this section concerning the CMININF problem and not in Section 9. The reason is that the complexity result for MININF immediately carries over to CMININF, but we do not want to present the same proof twice. We need first a supporting lemma, which explains how to force variables to take constant values.

**Lemma 8.17** *If  $S$  is bijunctive but not other Schaefer, then we can construct by conjunction and variable identification the binary relation  $R_{01} = \{01\}$ .*

**Proof:** The proof is almost the same as of Lemma 5.24 in [8]. Observe that  $S$  is neither 0-valid, nor 1-valid, nor complementive since this (according to Post's lattice) would imply that  $S$  is Horn, dual Horn, and affine, respectively.

Consider a non-0-valid relation  $R_{n0} \in S$ . If  $R_{n0}$  is 1-valid, then  $R_{n0}(x, \dots, x) = T(x)$ . Similarly if a non-1-valid relation  $R_{n1}$  is 0-valid then we can implement  $F(x)$  from  $R_{n1}$  by variable identification. Hence, we can implement  $R_{01}(x, y)$  by  $F(x) \wedge T(y)$ .

If  $R_{n0}$  is neither 0-valid nor complementive, nor 1-valid, then  $0 \cdots 0 \notin R_{n0}$ ,  $1 \cdots 1 \notin R_{n0}$ , and there exists a tuple  $s \in R_{n0}$  such that  $\neg s \notin R_{n0}$ , where  $\neg s$  denotes the complement of  $s$ . Define  $I_0(s) = \{i \mid s[i] = 0\}$  and  $I_1(s) = \{i \mid s[i] = 1\}$  and form the constraint  $R_{n0}(x_i, \dots, x_j)$ , where  $x_i, \dots, x_j \in \{x_0, x_1\}$ , by placing  $x_0$  at the positions in  $I_0(s)$  and  $x_1$  at the positions in  $I_1(s)$ . Note that  $I_0(s)$  and  $I_1(s)$  are both non-empty since  $R_{n0}$  is neither 0-valid nor 1-valid. Hence, we have  $R_{n0}(x_i, \dots, x_j) = R_{01}(x_0, x_1)$ .

Finally, if  $R_{n0} \in S$  is complementive but neither 0-valid nor 1-valid, then there must be a relation  $R_{nc} \in S$  which is not complementive, since a bijunctive and complementive relations is also affine, therefore also 0- and 1-valid. Hence, there exists a tuple  $s \in R_{nc}$  such that  $\neg s \notin R_{nc}$ . Form the constraint  $R_{n10}(x_0, x_1) = R_{nc}(x_i, \dots, x_j)$  (where  $x_i, \dots, x_j \in \{x_0, x_1\}$ ) by placing  $x_0$  at the positions in  $I_0(s)$  and  $x_1$  at the positions in  $I_1(s)$ . We have that  $01 \in R_{n10}$  and  $01 \notin R_{n10}$ . Consider again  $R_{n0}$  and a tuple  $s \in R_{n0}$ , then  $(x_0 \neq x_1) = R_{n0}(x_i, \dots, x_j)$  where  $x_i, \dots, x_j \in \{x_0, x_1\}$ ,  $x_0$  occurs at the positions in  $I_0(s)$ , and  $x_1$  at the positions in  $I_1(s)$ . We can again implement the constraint  $R_{01}(x_0, x_1) = (x_0 \neq x_1) \wedge R_{n10}(x_0, x_1)$ .  $\square$

**Proposition 8.18** *Let  $N = \{[\neg x \vee \neg y], [x \neq y]\}$ .  $\text{MININF}(S)$  for each set of relations  $S$ , which is bijunctive, but neither other Schaefer, nor a subset of  $\langle N \rangle_{\neq}$ , is coNP-complete.*

**Proof:** Let  $R$  be the Cartesian product of all relations in  $S$ . Obviously,  $R$  is bijunctive but neither Horn, nor dual Horn, nor affine, nor a subset of  $\langle N \rangle_{\neq}$ . Let  $\varphi$  be the conjunction of binary constraints representing the relation  $R$ , produced according to Proposition 8.16.

There must be two literals  $\ell_p$  and  $\ell$ , such that  $(\ell_p \vee \ell) \in \varphi$  and  $\ell_p$  being positive, otherwise we would have  $[\varphi] \in \langle N \rangle_{\neq}$ . Moreover,  $\ell_p$  must not occur in a  $\neq$  constraint, otherwise we would again have  $[\varphi] \in \langle N \rangle_{\neq}$ . This is because,  $(\ell_p \vee \ell) \wedge (\ell_p \neq x) = (\neg x \vee \ell) \wedge (\ell_p \neq x)$ .

Hence, we are in the situation  $(\ell_p \vee \ell) \in \varphi$  and  $\ell_p$  is positive and it does not occur in any  $\neq$  constraint. Two cases emerge depending on whether  $\ell$  appears in a  $\neq$  constraint. For the rest of the proof we focus on the case where  $\ell$  appears in a  $\neq$  constraint. The other case can be handled in the same way with only minor and obvious modifications. Assume that  $\ell_p = x$ ,  $\text{Var}(\ell) = y$ , and  $y$  occurs in a  $\neq$  constraint together with  $z$ , i.e., that  $(x \vee y) \wedge (y \neq z) \in \varphi$ , or  $(x \vee \neg y) \wedge (y \neq z) \in \varphi$ . Without loss of generality we assume the former case since  $(x \vee \neg y) \wedge (y \neq z) = (x \vee z) \wedge (y \neq z)$ .

Our goal is to produce the constraint  $(x \vee y) \wedge (y \neq z) \wedge R_{01}(v, w)$  from  $\varphi$  by conjunction and variable identification. It is clear that both aforementioned constraints belong to  $\langle S \rangle_{\neq}$ , since we can produce from  $S$  the constraint  $R_{01}(x_0, x_1)$  following Lemma 8.17.

Simplify  $\varphi$  by identifying all variables  $x_1, x_2$  such that  $(x_1 = x_2) \in \varphi$ , then remove all equality constraints from  $\varphi$ . Form the sets of variables  $V_0 = \{v \mid \varphi \models \neg v\}$  and  $V_1 = \{w \mid \varphi \models w\}$ . Remove from  $\varphi$  all clauses containing variables in  $V_0 \cup V_1$ , then add the constraint  $R_{01}(V_0, V_1)$  to  $\varphi$ . Obviously, this transformation of  $\varphi$  does not change the fact that the projection of  $\varphi$  onto  $\{x, y\}$  is equivalent to  $(x \vee y)$ , whereas the projection of  $\varphi$  onto  $\{y, z\}$  is equivalent to  $(y \neq z)$ . Note that there can be no  $(x_1 = x_2) \in \varphi$  where  $\{x_1, x_2\} \subseteq \{x, y, z\}$ .

Continue simplifying the formula  $\varphi$  by eliminating all  $(x_1 \neq x_2)$  constraints, except the one  $(y \neq z)$ , by replacing  $x_1$  by  $\neg x_2$ , as well as  $\neg x_1$  by  $x_2$ , throughout the formula. Denote the resulting formula by  $\varphi'$ . Since  $x$  does not occur in any  $\neq$  constraint and neither  $y$  nor  $z$  occur in any other  $\neq$  constraint except in  $(y \neq z)$ , otherwise there would have to be some  $(x_1 = x_2)$  constraint still left, we have  $(x \vee y) \wedge (y \neq z) \in \varphi'$ .

Recall that resolution between two clauses  $(c \vee v)$  and  $(\neg v \vee c')$  produces the new clause  $(c \vee c')$  and discards the two previous clauses. Note that resolution on binary clauses produces a binary clause. For any variable  $x_i \notin \{x, y, z\}$  occurring both positively and negatively in  $\varphi'$ , apply resolution to get a formula, where every variable not in  $\{x, y, z\}$  only occurs either positively or negatively, but not both. Denote the resulting formula by  $\varphi''$ . Form the variable sets  $V_0'' = \{v \mid v \text{ only occurs negatively in } \varphi''\}$  and  $V_1'' = \{w \mid w \text{ only occurs positively in } \varphi''\}$ . Discard from  $\varphi''$  the clauses containing variables from  $V_0'' \cup V_1''$  and add the constraint  $R_{01}(V_0'', V_1'')$  to the formula.

Hence using only conjunction and variable identification, we can implement the constraint  $(x \vee y) \wedge (y \neq z) \wedge R_{01}(v, w)$  from  $S$ . Using Lemma 8.15 we obtain the desired coNP-completeness result.  $\square$

## 8.6 Main Result

We put together all results in this section and obtain the following trichotomy result for minimal inference without free variables but with constants.

**Theorem 8.19 (Trichotomy of CMININF)** *Let  $S$  be a finite nonempty set of Boolean relations and  $S^*$  the corresponding 1-valid restriction. If every relation in  $S$  is both Horn and dual Horn, or negative Horn, or Schaefer and incomparable, or a subset of the weak co-clone  $\langle N \rangle_{\neq}$  (where  $N = \{\{\neg x \vee \neg y\}, \{x \neq y\}\}$ ), then  $\text{CMININF}(S)$  is decidable in polynomial time. Else if  $S^*$  is Schaefer, then  $\text{CMININF}(S)$  is coNP-complete. Otherwise,  $\text{CMININF}(S)$  is  $\Pi_2\text{P}$ -complete.*

**Proof:** The parts concerning  $\Pi_2\text{P}$ -completeness and membership in coNP follow from the dichotomy theorem due to Kirousis and Kolaitis [18]. As for tractability, the fact that  $\text{CMININF}(S)$  is tractable when every relation in  $S$  is both Horn and dual Horn, or negative Horn, follows directly from the corresponding tractability results for  $\text{GMININF}$  proved in [6], and in Proposition 5.4, respectively. Tractability of  $\text{MININF}(S)$  for  $S$  being Schaefer and incomparable, or a subset of  $\langle N \rangle_{\neq}$ , is proved in Propositions 8.4, and 8.6, respectively. The coNP-hardness for  $\text{CMININF}(S)$  when  $S$  is not Schaefer, and  $S^*$  is Schaefer, is proved in Proposition 8.2.

Hence, what remains to be done is to prove coNP-hardness for all sets of relations  $S$  that do not fall into one of the tractable classes when  $S$  is dual Horn, Horn, affine, or bijunctive. This is

done in Sections 8.2, 8.3, 8.4, and 8.5, respectively.  $\square$

## 9 Minimal Inference Without Free Variables Nor Constants

We want to derive in this section a complete characterization of complexity for the MININF problem, which is the most challenging among the considered minimal inference problems. It requires *all* variables of  $\varphi$  in a MININF instance  $\varphi \models_{\min} c$  to be minimized. The difficulty of the problem originates in the impossibility to use the Galois correspondence or the possibility to substitute logical constants for variables in the relations. In fact, MININF is the most difficult problem to analyse among all four cases of minimal inference.

Many results for MININF were already presented in previous sections, since they represent necessary stepping stones to prove other complexity results for the three previously considered variants of minimal inference. Recall also that MININF is a special case of GMININF, VMININF, or CMININF with the supplementary condition  $Q = \emptyset$  and  $Z = \emptyset$ . Hence a tractability result for the more general case of minimal inference, namely Proposition 5.3, Proposition 5.4, Proposition 5.5, Proposition 6.15, Proposition 6.16, and Proposition 8.4, propagates.

We use the same two phase approach as in the previous section, considering first non-Schaefer relations  $S$ , where we proceed with a sharpening of a result from Kirousis and Kolaitis [18].

**Proposition 9.1** *Let  $S$  be a non-Schaefer and non-0-valid set of Boolean relations, with  $S^*$  being the corresponding 1-valid restriction. If  $S^*$  is Schaefer, then  $\text{MININF}(S)$  is coNP-complete, otherwise it is  $\Pi_2\text{P}$ -complete.*

**Proof:** The  $\Pi_2\text{P}$ -completeness part and the fact that  $\text{MININF}(S)$  is in coNP when  $S^*$  is Schaefer is proved in [18, Theorem 3.10]. Hence, what remains to be proved is that  $\text{MININF}(S)$  is coNP-hard when  $S^*$  is Schaefer. First note that the case where  $S$  is non-Schaefer and non-0-valid but 1-valid is known to be  $\Pi_2\text{P}$ -complete by Theorem 3.10 in [18], hence we can assume that  $S$  is non-1-valid. We give a reduction from the coNP-complete unsatisfiability problem  $\text{UNSAT}(S)$  to  $\text{MININF}(S)$ . Let  $\varphi$  be a conjunction of constraints over  $S$ . Let  $R(x_1, x_2, \dots, x_n)$  be a constraint build over a relation  $R$  in  $S$  where the variables  $x_1, \dots, x_n$  do not occur in  $\varphi$ . Let  $m$  be a minimal model of  $R(x_1, x_2, \dots, x_n)$ . We construct the clause to be inferred  $\psi$  as follows: If  $m(x_i) = 1$  ( $m(x_i) = 0$ ) then add  $\neg x_i$  ( $x_i$ ) to  $\psi$  for  $1 \leq i \leq n$ . Then  $\varphi$  is unsatisfiable if and only if  $\psi$  can be minimally inferred from  $\varphi \wedge R(x_1, x_2, \dots, x_n)$ .  $\square$

Consider now  $S$  to be Schaefer or 0-valid. Tractability of  $\text{MININF}(S)$  for  $S$  being 0-valid is trivial and the polynomial-time decidability of  $\text{MININF}(S)$  for  $S$  being Horn follows from the fact that a satisfiable Horn formula has a unique minimal model computable in polynomial time. In the rest of the paper we investigate the other cases for which  $\text{MININF}(S)$  is in coNP, namely when  $S$  is dual Horn, bijunctive, or affine.

### 9.1 Affine Relations

We need first a representation lemma proved by Schaefer's technique.

**Lemma 9.2** *Let  $R$  be a Boolean relation which is affine and 1-valid, but neither 0-valid, nor other Schaefer. Then we can construct the relation  $[(x + y + z = 1) \wedge (w = 1)]$  from  $R$  by conjunction and variable identification.*

**Proof:** Since  $R$  is not Horn, there exist two vectors  $a, b \in R$ , such that  $a \wedge b \notin R$ . Construct the variable sets  $V_{ij} = \{x \in V \mid a(x) = i, b(x) = j\}$ . Identify the variables in each set  $V_{ij}$  and construct the relation  $R'$  generated by the constraint  $R[V_{00}, V_{01}, V_{10}, V_{11}]$ . It can be easily seen that the vectors 0011, 0101, and 1111 belong to  $R'$ , whereas 0000  $\notin R'$  (not 0-valid) and 0001  $\notin R'$  (not Horn). Since  $R'$  is affine, we have  $0011 + 0101 + 1111 = 1001 \in R'$ .

Let  $m, m' \in R'$  and  $m'' \notin R'$ . Then also  $m + m' + m'' \notin R'$ , since otherwise from the membership of  $m, m'$ , and  $m + m' + m''$  in  $R'$  follows that  $(m + m' + m'') + m + m' = m'' \in R'$ , because  $R'$  is affine. Using this result, from  $0011, 0101 \in R'$  and  $0001 \notin R'$  follows  $0110 \notin R'$ . From  $0101, 1111 \in R'$  and  $0001 \notin R'$  follows  $1011 \notin R'$ . Finally from  $0011, 1111 \in R'$  and  $0001 \notin R'$  follows  $1101 \notin R'$ . We can force the variable  $V_{11}$  to take the value 1 by the constraint  $R'(V_{11}, V_{11}, V_{11}, V_{11})$ , since  $R$  as well as  $R'$  are both 1-valid but not 0-valid. Hence, the constraint  $R'(x, y, z, w) \wedge R'(w, w, w, w)$  generates the relation  $[(x + y + z = 1) \wedge (w = 1)] = \{0011, 0101, 1001, 1111\}$ .  $\square$

**Proposition 9.3**  $\text{MININF}(S)$  is coNP-complete for each set of relations  $S$  which is affine and 1-valid, but neither 0-valid, nor other Schaefer.

**Proof:** We can construct a relation  $R$  which is affine and 1-valid, but not 0-valid, not Horn, not dual Horn, nor bijunctive, by Cartesian product of relations from  $S$ . The rest follows from Lemma 9.2, Lemma 6.13, and from the relationship between MINEXT and MININF.  $\square$

We need a supporting lemma explaining in the context of affine constraint languages how to force variables to take constant values.

**Lemma 9.4** If  $S$  is neither complementive nor 0-valid nor 1-valid, then the relation  $R_{01} = \{01\}$  can be constructed from  $S$  by conjunction and variable identification.

**Proof:** We can assume the existence of a relation  $R$  which is neither complementive nor 0-valid nor 1-valid by taking Cartesian products of relations in  $S$  according to Lemma 7.1. Take a tuple  $a \in R$  such that  $\neg a \notin R$  (since  $R$  is not complementive) and perform Schaefer's construction on  $a$ , i.e., construct the relation  $Q$  generated by the constraint  $R[V_0, V_1]$ . Obviously  $01 \in Q$  holds, and neither  $00$  nor  $11$  is in  $Q$  (because  $R$  is neither 0-valid nor 1-valid). Moreover  $10 \notin Q$  holds since  $\neg a \notin R$ . Thus we have constructed the desired relation  $Q = \{01\}$ .  $\square$

**Proposition 9.5**  $\text{MININF}(S)$  is coNP-hard for each set of relations  $S$ , which is affine, but neither incomparable, nor other Schaefer, nor 0-valid, nor 1-valid, nor complementive.

**Proof:** By Lemma 9.4 we know that we have access to the constraint  $R_{01}(x, y)$ , based on the relation  $R_{01} = \{01\}$ , forcing the variables  $x$  and  $y$  to take constant values. Since  $S$  is neither Horn nor dual Horn, by taking Cartesian products we can assume that there is a relation  $N \in \langle S \rangle$  and two tuples  $a, b \in N$ , such that  $a \wedge b \notin N$  and  $a \vee b \notin N$ . Use Schaefer's variable identification on  $a$  and  $b$  to get the relation  $N[V_{00}, V_{01}, V_{10}, V_{11}]$ . Now we need to impose the constant 0 on  $V_{00}$  and 1 on  $V_{11}$ , using the relation  $R_{01}$ . We construct the constraint  $N(z, x, y, w) \wedge R_{01}(z, w) = (x \neq y) \wedge (z = 0) \wedge (w = 1)$  by conjunction and variable identification. The rest of the proof is totally identical to that of Proposition 8.14.  $\square$

We need a representation lemma for complementive relations and a complexity result for a minimal inference problem upon a particular relation  $R$  which is both affine and complementive.

**Lemma 9.6** *If  $S$  is complementive and neither 0-valid nor 1-valid, then we can construct  $[x \neq y]$  from  $S$  by conjunction and variable identification.*

**Proof:** We can assume the existence of a relation  $R$  by taking Cartesian products according to Lemma 7.1, such that  $R$  is neither 0-valid nor 1-valid. Take two tuples  $a, b \in R$  such that  $\neg a = b$ . Perform Schaefer's construction  $R[V_{00}, V_{01}, V_{10}, V_{11}]$  on  $a$  and  $b$ . Obviously the variables  $V_{00}$  and  $V_{11}$  do not appear since  $a$  is the complement of  $b$ . Hence we have the binary relation  $R[V_{01}, V_{10}]$  containing the tuples 01 and 10, but not the tuples 11 and 00 (since otherwise  $R$  would be 1-valid and 0-valid, respectively). Hence the relation generated by the constraint  $R[V_{01}, V_{10}]$  is  $\{01, 10\} = [x \neq y]$ .  $\square$

**Lemma 9.7**  $\text{MINEXT}(R)$  is NP-complete for  $R = [(x + y + z + w = 0) \wedge (x + u = 1) \wedge (y + v = 1)]$ .

**Proof:** NP-membership is clear, we focus on the NP-hardness proof by a reduction from NAE3SAT (not-all-equal 3SAT). Let  $\varphi = c_1 \wedge \dots \wedge c_k$  be a not-all-equal 3SAT formula with variables  $x_1, \dots, x_n$  and clauses  $c_i = l_i^1 \vee l_i^2 \vee l_i^3$ . We construct a formula  $\varphi'$  as the conjunction of the following equations for each clause  $c_i = l_i^1 \vee l_i^2 \vee l_i^3$ :

$$\begin{aligned} (z_{3i-2} + v_i^1 + v_i^2 + y_i = 0) \wedge (v_i^1 + u_i^1 = 1) \wedge (v_i^2 + u_i^2 = 1), \\ (z_{3i-1} + v_i^2 + v_i^3 + y_i = 0) \wedge (v_i^2 + u_i^2 = 1) \wedge (v_i^3 + u_i^3 = 1), \\ (z_{3i} + v_i^3 + v_i^1 + y_i = 0) \wedge (v_i^3 + u_i^3 = 1) \wedge (v_i^1 + u_i^1 = 1). \end{aligned}$$

where

$$v_i^j = \begin{cases} x_p & \text{if } l_i^j = x_p, \\ x'_p & \text{if } l_i^j = \neg x_p. \end{cases} \quad \text{and} \quad u_i^j = \begin{cases} x'_p & \text{if } v_i^j = x_p, \\ x_p & \text{if } v_i^j = x'_p. \end{cases}$$

The variable  $v_i^j$  is a placeholder for the literal  $l_i^j$ , whereas  $u_i^j$  is the negation of  $v_i^j$ .

**Claim:** The formula  $\varphi$  is satisfiable if and only if the assignment  $s(y_i) = 1$  for all  $i = 1, \dots, k$  of  $\varphi'$  has a minimal extension.

Let  $\varphi$  be not-all-equal satisfiable. Then there exists a not-all-equal satisfying assignment  $m$  of the formula  $\varphi$ . Since every clause  $c_i$  is not-all-equal satisfied, for each  $i$  there must be  $a, b \in \{1, 2, 3\}$  such that  $m(l_i^a) \neq m(l_i^b)$ . Let  $\bar{m}$  be an extension of  $m$  that satisfies  $\varphi'$ . Following the construction of  $\varphi'$ , we must have  $\bar{m}(v_i^a) + \bar{m}(v_i^b) = 1$ . Then there are two possibilities to get a minimal assignment  $\bar{m}$ :

1. When we set  $\bar{m}(z_{3i-p}) = 1$  and  $\bar{m}(y_i) = 0$ , we get an uninteresting assignment.
2. When we set  $\bar{m}(z_{3i-p}) = 0$  and  $\bar{m}(y_i) = 1$ , we get an assignment which is an extension of the assignment  $s(y_i) = 1$  for all  $i = 1, \dots, k$ .

These two possible assignments are clearly incomparable and no value can be changed from 1 to 0 to get another satisfying assignment of  $\varphi'$ . Therefore the assignment  $\bar{m}$  from the second case is minimal.

Let  $\varphi$  be not-all-equal unsatisfiable. Then for each assignment  $m$  there must always be a clause  $c_i$  which literals are assigned the same values, i.e.,  $m(l_i^1) = m(l_i^2) = m(l_i^3) = 0$  or  $m(l_i^1) = m(l_i^2) = m(l_i^3) = 1$ . Let  $\bar{m}$  be an extension of  $m$  that satisfies the formula  $\varphi'$ . Following the construction of  $\varphi'$ , we have  $\bar{m}(v_i^1) + \bar{m}(v_i^2) = \bar{m}(v_i^2) + \bar{m}(v_i^3) = \bar{m}(v_i^3) + \bar{m}(v_i^1) = 0$ . Then we can set  $\bar{m}(z_{3i-2}) = \bar{m}(z_{3i-1}) = \bar{m}(z_{3i}) = \bar{m}(y_i) = 0$  to produce a minimal assignment. This implies that the assignment  $s(y_i) = 1$  for all  $i$  cannot be extended to a minimal one.  $\square$

**Proposition 9.8**  $\text{MININF}(S)$  is coNP-hard for each set of relations  $S$ , which is affine and complementive, but neither incomparable, nor other Schaefer, nor 0-valid, nor 1-valid.

**Proof:** By taking Cartesian products of relations in  $S$  there exists a relation  $R$  in  $\langle S \rangle_{\#}$ , according to Lemma 7.1, which is affine, but neither incomparable, nor bijunctive, nor Horn, nor dual-Horn. Since  $R$  is not incomparable there are two tuples  $a$  and  $b$  satisfying the condition  $a < b$ . Without loss of generality we can assume that  $a$  and  $b$  are closest, i.e., there is no tuple  $t$  satisfying  $a < t < b$ . By Lemma 8.13 we know that  $a$  and  $b$  differ in at least two positions and that there exists a third tuple  $c$  which is not constant on the coordinates where  $a$  and  $b$  differ.

Since  $R$  is neither Horn, nor dual-Horn we can assume, by taking a Cartesian products of  $R$  with itself, that  $c$  satisfies the conditions  $a \wedge c \notin R$ ,  $b \wedge c \notin R$ ,  $a \vee c \notin R$ , and  $b \vee c \notin R$ . Form the Schaefer style implementation on  $R$  based on the tuples  $a, b, c$ . Note that since  $a < b$  holds, the variables  $V_{100}$  and  $V_{101}$  will not appear. Hence we get the relation  $R[V_{000}, V_{001}, V_{010}, V_{011}, V_{110}, V_{111}]$ . It can be checked that the variables  $V_{001}, V_{010}, V_{011}, V_{110}$  must all appear. If  $V_{001}$  does not appear, then  $b \vee c = b \in R$  which is a contradiction. Similarly, if  $V_{110}$  does not appear, then  $a \wedge c = a \in R$  which is a contradiction. Moreover,  $V_{010}$  and  $V_{011}$  must appear since  $c$  is not constant on the coordinates where  $a$  and  $b$  differ.

Since  $R$  is affine, it must also contain the tuple  $d = a + b + c = 011001$ . Moreover,  $R$  does not contain the tuple  $t = \text{maj}(a, b, c) = 000111$  since this tuple satisfies  $a < t < b$ . Furthermore, since  $R$  is complementive it also contains the tuples 100110, 111100 101010, 110000. Now, adding the constraints  $V_{000} \neq V_{111}$  and  $V_{110} \neq V_{001}$ , to which we have access according to Lemma 9.6, we get the relation  $R[V_{000}, V_{001}, V_{010}, V_{011}, V_{110}, V_{111}] \wedge (V_{000} \neq V_{111}) \wedge (V_{110} \neq V_{001})$ . Since the value of  $V_{110}$  is determined by  $V_{001}$  and vice versa, and the same for  $V_{000}$  and  $V_{111}$ , there can be at most 16 tuples satisfying the affine constraint. We already have 8 tuples that satisfy the constraint and we know that the tuple 000111 does not satisfy the constraint. Since the constraint is affine, the number of tuples satisfying the constraint must be a power of 2. The constraint  $R[V_{000}, V_{001}, V_{010}, V_{011}, V_{110}, V_{111}] \wedge (V_{000} \neq V_{111}) \wedge (V_{110} \neq V_{001})$  generates the relation consisting of the tuples  $\{000011, 001111, 010101, 011001, 100110, 111100, 101010, 110000\}$ . It is equivalent to the constraint  $(V_{000} + V_{001} + V_{010} + V_{011} = 0) \wedge (V_{000} + V_{111} = 1) \wedge (V_{001} + V_{110} = 1)$ , for which the minimal inference problem is coNP-complete according to Lemma 9.7.  $\square$

## 9.2 Dual Horn Relations

**Proposition 9.9**  $\text{MININF}(S)$  is coNP-hard for each set of relations  $S$ , which is dual Horn, but neither Horn, nor 0-valid, nor 1-valid.

**Proof:** Construct a relation  $R$  from  $S$  which is neither 0-valid nor 1-valid. Such a relation exists by taking Cartesian products. Take a tuple  $a \in R$  and construct the relation  $R[V_0, V_1]$  on  $a$ . Both variables  $V_0$  and  $V_1$  must appear since  $R$  is neither 0-valid nor 1-valid. Hence  $01 \in R[V_0, V_1]$ . Neither  $00$  nor  $11$  is in  $R[V_0, V_1]$  since  $R$  is neither 0-valid nor 1-valid. Moreover,  $10$  is not in  $R$ , since if it were then the fact that  $R$  is dual-Horn together with  $01 \in R$  implies  $11 \in R$  which we have already ruled out. Thus  $R[V_0, V_1]$  represents the relation  $\{01\}$ .

Now construct from  $S$  a relation  $Q$  which is dual Horn, but not Horn. There must be two tuples  $a, b \in Q$ , such that  $a \wedge b \notin Q$  and  $a \vee b \in Q$ . Construct the relation  $Q[V_{00}, V_{01}, V_{10}, V_{11}]$  on  $a$  and  $b$ . Form the conjunction  $Q[V_{00}, V_{01}, V_{10}, V_{11}] \wedge R[V_{00}, V_{11}]$  forcing the variables  $V_{00}$  and  $V_{11}$  to take the values 0 and 1, respectively. The constraint  $Q'(V_{00}, V_{01}, V_{10}, V_{11}) = Q[V_{00}, V_{01}, V_{10}, V_{11}] \wedge R[V_{00}, V_{11}]$

is equal to  $(V_{01} \vee V_{10}) \wedge (V_{00} = 0) \wedge (V_{11} = 1)$ . Cadoli and Lenzerini [6, Theorem 5] proved that  $\text{MININF}([x \vee y])$  is coNP-complete, therefore also  $\text{MININF}(Q')$  where  $Q' = [(x \vee y) \wedge (z = 0) \wedge (w = 1)]$  is coNP-complete.  $\square$

**Lemma 9.10**  $\text{MININF}(R)$  is coNP-complete for  $R = [(x \vee y) \wedge (\neg z \vee x) \wedge (\neg z \vee y)]$  and  $R = [(x \vee y) \wedge (\neg z \vee x)]$ .

**Proof:** The membership is clear, we focus on the hardness proof done by a simple reduction from  $\text{MININF}([x \vee y])$ . Every constraint  $(x \vee y)$  in the  $\text{MININF}([x \vee y])$  instance is replaced by a constraint  $(x \vee y) \wedge (\neg z_i \vee x) \wedge (\neg z_i \vee y)$  or  $(x \vee y) \wedge (\neg z_i \vee x)$ , respectively, where the  $z_i$ 's are fresh variables (one for each new constraint). Note that all the  $z_i$ 's must be 0 in all minimal models of the new instance. Hence, there is an obvious one-to-one correspondence between the minimal models of the old and new  $\text{MININF}$  instances.  $\square$

**Proposition 9.11**  $\text{MININF}(S)$  is coNP-hard for each set of relations  $S$ , which is dual-Horn and 1-valid, but neither 0-valid nor Horn.

**Proof:** We know that there is a relation  $R$  in  $\langle S \rangle_{\#}$  which is dual-Horn and 1-valid but not 0-valid, constructed by taking Cartesian products. Now, do Schaefer's construction on two tuples  $a, b \in R$  such that  $a \wedge b \notin R$  and  $a \vee b \in R$ . Such tuples exist, since  $R$  is dual Horn, but not Horn. This construction gives us the relation  $Q$  generated by the constraint  $R[V_{00}, V_{01}, V_{10}, V_{11}]$ . Since  $Q$  is 1-valid but not 0-valid, we have access to the constraint  $(x = 1)$ . Construct the conjunction  $Q'(V_{00}, V_{01}, V_{10}, V_{11}) = Q(V_{00}, V_{01}, V_{10}, V_{11}) \wedge Q(V_{11}, V_{11}, V_{11}, V_{11})$ . Now,  $V_{01}$  and  $V_{10}$  both exists, otherwise we would have  $a \wedge b \in R$ . Moreover,  $0001 \notin Q'$  since  $a \wedge b = 0001$ . If  $V_{00}$  does not exist then  $Q' = [(x \vee y) \wedge (z = 1)]$ . Cadoli and Lenzerini proved in [6] that  $\text{MININF}([x \vee y])$  is coNP-complete, hence also  $\text{MININF}(Q')$  is coNP-complete.

Assume now that  $V_{00}$  exists. We know that  $Q'$  contains the tuples 1111, 0011, 0101, and 0111. We now have to consider 5 cases according to the presence of the tuples 1001, 1011, and 1101:

**Case 1:**  $Q' = \{1111, 0011, 0101, 0111\}$  which is equal to  $[(y \vee z) \wedge (\neg x \vee y) \wedge (\neg x \vee z) \wedge (w = 1)]$  for which  $\text{MININF}$  is coNP-complete according to Lemma 9.10.

**Case 2:**  $Q'$  contains the tuple 1001. Because  $Q'$  is dual Horn, it must also contain the tuples 1011 and 1101. Since 0001 is not in the relation, we have constructed  $[(x \vee y \vee z) \wedge (w = 1)]$ . By variable identification we can get the relation  $[(x \vee y) \wedge (w = 1)]$  for which we already know  $\text{MININF}$  to be coNP-complete. Hence we can assume further on that  $Q'$  does not contain the tuple 1001.

**Case 3:**  $Q'$  contains 1011. Then it is equal to  $[(y \vee z) \wedge (\neg x \vee z) \wedge (w = 1)]$  for which  $\text{MININF}$  is coNP-complete according to Lemma 9.10.

**Case 4:**  $Q'$  contains 1101. Then it is equal to  $[(y \vee z) \wedge (\neg x \vee y) \wedge (w = 1)]$  for which  $\text{MININF}$  is coNP-complete according to Lemma 9.10.

**Case 5:**  $Q'$  contains both tuples 1101 and 1011. Then we can construct the new constraint  $Q''(x, y, z, w) = Q'(x, y, z, w) \wedge Q(x, x, x, x)$  which is equivalent to  $(y \vee z) \wedge (x = 1) \wedge (w = 1)$ . Since we know that  $\text{MININF}([x \vee y])$  is coNP-complete, we also know that  $\text{MININF}(Q'')$  is coNP-complete.

There are no more cases to consider, what terminates the proof.  $\square$

### 9.3 Main Result

Putting together all previous propositions, we obtain the following trichotomy result for the minimal inference problem.

**Theorem 9.12 (Trichotomy of MININF)** *Let  $S$  be a finite nonempty set of Boolean relations and  $S^*$  the corresponding 1-valid restriction. If every relation in  $S$  is Horn, or 0-valid, or both Schaefer and incomparable, or a subset of the weak co-clone  $\langle N \rangle_{\#}$  (where  $N = \{\neg x \vee \neg y, [x \neq y]\}$ ), then  $\text{MININF}(S)$  is decidable in polynomial time. Else if  $S^*$  is Schaefer, then  $\text{MININF}(S)$  is coNP-complete. Otherwise,  $\text{MININF}(S)$  is  $\Pi_2\text{P}$ -complete.*

**Proof:** The parts concerning  $\Pi_2\text{P}$ -completeness and membership in coNP follow from the dichotomy theorem due to Kirousis and Kolaitis [18]. As for tractability, this is, as already explained, trivial for  $S$  being Horn or 0-valid. Tractability of  $\text{MININF}(S)$  for  $S$  being Schaefer and incomparable, or a subset of  $\langle N \rangle_{\#}$ , is proved in Propositions 8.4, and 8.6, respectively. The coNP-hardness for  $\text{MININF}(S)$  when  $S$  is neither Schaefer, nor 0-valid, and  $S^*$  is Schaefer, is proved in Proposition 9.1.

Hence, what remains to be done is to prove coNP-hardness for all sets of relations  $S$  that do not fall into one of the tractable classes when  $S$  is affine, dual Horn, or bijunctive. This is done in Sections 9.1, 9.2, and 8.5, respectively. For  $S$  being affine, the analysis is divided into three cases, depending on whether  $S$  is 1-valid (Proposition 9.3), complementive (Proposition 9.8), or neither 1-valid nor complementive (Proposition 9.5). Similarly, the analysis for the dual Horn case is divided into two parts depending on whether  $S$  is 1-valid (Proposition 9.11) or not (Proposition 9.9). Finally, the case where  $S$  is bijunctive is treated in Proposition 8.18.  $\square$

## 10 Concluding Remarks

We have proved trichotomy theorems for the complexity of four variants of the generalized minimal inference problem. This also covers the original minimal inference problem, where all variables of the formula  $\varphi$  are being minimized. A careful reader certainly discovered that we rely on many previous results, previously published by Eiter and Gottlob [12], Cadoli and Lenzerini [6], Kirousis and Kolaitis [18], and finally by Nordh and Jonsson [21].

Our trichotomy result for  $\text{MININF}$ , presented in Theorem 9.12, finally settles with a positive answer the trichotomy conjecture of Kirousis and Kolaitis formulated in [18] and also appearing as one of the open problems (Question 4.1) during the International Workshop on Mathematics of Constraint Satisfaction held in Oxford in March 2006 [1]. In the process we also strengthen and give a much simplified proof of the main result from [11], which concerns the coNP-hardness of  $\text{MININF}$  over affine relations.

Our proof techniques for the two variants with free variables (i.e.,  $\text{GMININF}$  and  $\text{VMININF}$ ) are based on the well known and powerful algebraic approach via Post's lattice of co-clones. For the two variants without free variables (i.e.,  $\text{CMININF}$  and  $\text{MININF}$ ) this approach provably does not work since the complexity borderlines cuts through the co-clones. For these problems we instead refine Schaefer's approach via a particular type of *implementations*. There is one significant difference between Schaefer's implementations and ours, namely we only use conjunction, variable identification, and variable permutation (*no* existential quantification) in our implementations. The reason for not allowing existential quantification in our implementations is that the structure of minimal models of a formula is destroyed by existentially quantifying some variables. Moreover as manifested by

both the results in this paper and the results due to Kirousis and Kolaitis [18], the complexity of CMININF and MININF is not preserved by implementations using existential quantification.

Our implementations could be interesting in their own right since they might be useful in classifying the complexity of other problems where existential quantification does not preserve the complexity. Note that Schaefer's original implementations have indeed been reused many times and for many different problems [8].

Another observation that we make is that our coNP-hardness proofs for CMININF and MININF seems to require that the clause (query) to be minimally inferred can be arbitrary long (i.e., its length is allowed to depend on the size of the knowledge base). This is in sharp contrast with the  $\Pi_2$ P-hardness proofs for CMININF and MININF proved by Kirousis and Kolaitis [18], where they observe that a  $\Pi_2$ P-hardness holds even when the clause to be inferred consists of two literals. It would be interesting to investigate whether we could enlarge the tractable classes for minimal inference by bounding the length of the clause to be inferred by a constant  $k$ . This restriction is certainly relevant from a practical perspective since in the typical situation the knowledge base is very large in comparison to the clause to be inferred (i.e., the query).

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