A Linear-Programming Approach to Temporal Reasoning

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Abstract

We present a new formalism, Horn Disjunctive Linear Relations (Horn DLRs), for reasoning about temporal constraints. We prove that deciding satisfiability of sets of Horn DLRs is polynomial by exhibiting an algorithm based upon linear programming. Furthermore, we prove that most other approaches to tractable temporal constraint reasoning can be encoded as Horn DLRs, including the ORD-Horn algebra and most methods for purely quantitative reasoning.

Introduction

Reasoning about temporal constraints is an important task in many areas of AI and elsewehere, such as planning, natural language processing, diagnosis, time serialization in archeology *etc.* In most applications, knowledge of temporal constraints is expressed in terms of collections of relations between time intervals or time points. Typical reasoning tasks include determining the satisfiability of such collections and deducing new relations from those that are known. The research has largely concentrated on two kinds of formalisms; systems of inequalities on time points (Dechter, Meiri, & Pearl 1991; Meiri 1991; Koubarakis 1992) to encode quantitative information, and systems of constraints in Allen's algebra (Allen 1983) to encode qualitative relations between time intervals. Some attempts have been made to integrate quantitative and qualitative reasoning into unified frameworks (Kautz & Ladkin 1991; Meiri 1991). Since the satisfiability problem is NP-complete for Allen's algebra the qualitative and unified approaches have suffered from computational difficulties.

In response to the computational hardness of the full Allen algebra, several polynomial subalgebras have been proposed in the literature (van Beek & Cohen 1990; Golumbic & Shamir 1993; Nebel & Bürckert 1995). Some of these algebras have later been extended with mechanisms for handling quantitative information. For example the TimeGraph II system (Gerevini, Schubert, & Schaeffer 1993) extends the *pointisable algebra* (van Beek & Cohen 1990) with a limited type of quantitative information. Of special interest is the ORD-Horn algebra (Nebel & Bürckert 1995) which, under certain conditions, is the *unique* maximal tractable subclass of Allen's algebra. Hence, it would be especially interesting to extend this algebra with quantitative information since the maximality result would carry over to the new algebra, at least with respect to its qualitative expressiveness. To our knowledge, no such attempt have been made.

Now, to make the topic of reasoning about temporal constraint more concrete, consider the following fictious crime scenario. Professor Jones has been found shot on the beach near her house. Rumours tell that she was almost sure of having a proof that P≠NP, but had not yet shown it to any of her colleagues. The graduate student Hill is soon to defend his thesis on his newly invented complexity class, $\mathrm{NRQP}_{\Sigma}(\eth)^{✠}$, which would unfortunately be of no value were it to be known for certain that P≠NP. Needless to say, Hill is thus one of the prime suspects and inspector Smith is faced with the following facts and observations:

- Professor Jones died between 6 pm and 11 pm, according to the post-mortem.
- Mr Green, who lives close to the beach, is certain that he heard a gunshot sometime in the evening, but certainly after the TV news.
- The TV news is from 7.30 pm to 8.00 pm.
- A reliable neighbour of Hill claims Hill arrived at home sometime between 9.15 pm and 9.30 pm.
- It takes between 10 and 20 mins. to walk or run from the place of the crime to the closest parking lot.
- It takes between 45 and 60 mins. to drive from this parking lot to Hill's home.

The first thing to do is verifying that these facts and observations are consistent, which is obviously the case here. We can also draw some further conclusions, like narrowing the time of death to the interval between 8.00 pm and 11 pm, assuming the gunshot heard by mr Green actually was the killing shot.

Now, suppose inspector Smith adds the hypothesis that Hill was at the place of the murder at the time of the gunshot, which is only known to occur somewhere in the interval from 8.00 pm to 11.00 pm. If the set of facts and observations together with this hypothesis becomes inconsistent, then inspector Smith can rule out Hill as the murderer[1].

This problem can easily be cast in terms of a temporal-constraint-reasoning problem, involving both quantitative and qualitative relations over time points, intervals and durations. Unfortunately, it seems like this simple example cannot be solved by any of the computationally tractable methods reported in the literature. It can, however, be solved in polynomial time by the method proposed in this paper.

We introduce *Horn Disjunctive Linear Relations* (Horn DLRs for short) which is a temporal constraint formalism that allows for polynomial-time satisfiability checking. Horn DLRs subsumes the ORD-Horn algebra *and* most of the formalisms for encoding quantitative information proposed in the literature. The approach is rather different from the commonly used constraint network or graph-theoretic approaches. We base our method upon linear programming which proves to be a convenient tool for managing temporal information. Since most of the low-level handling of time points is thus abstracted away, the resulting algorithm is surprisingly simple. We strongly believe that Horn DLRs are useful in other areas of computer science than temporal reasoning. For instance, the proposal for constraint query languages in deductive databases by Kanellakis *et al.* (1995) has some resemblance with Horn DLRs.

The paper is structured as follows. We begin by giving basic terminology and definitions used in the rest of the paper together with a brief introduction to complexity issues in linear programming. We continue by presenting the polynomial-time algorithm for deciding satisfiability of Horn DLRs. The paper concludes with a short discussion of the results.

Disjunctive Linear Relations

We begin by defining different types of linear relations.

Definition 1 Let $X = \{x_1, \ldots, x_n\}$ be a set of real-valued variables. Let α, β be linear polynomials (*i.e.* polynomials of degree one) over X. A *linear disequation* over X is an expression of the form $\alpha \neq \beta$. A *linear equality* over X is an expression of the form $\alpha = \beta$. A *linear relation* over X is an expression of the form $\alpha r \beta$ where $r \in \{<, \leq, =, \neq, \geq, >\}$. A *convex linear relation* over X is an expression of the form $\alpha r_c \beta$ where $r_c \in \{<, \leq, =, \geq, >\}$. A *disequational linear relation* over X is an expression of the form $\alpha \neq \beta$. A *disjunctive linear relation* (DLR) is a disjunction of one or more linear relations.

Example 2 A typical DLR over $\{x_1, x_2, x_3\}$ is $(1.2x_1 + x_2 \leq x_3 + 5) \vee (12x_3 \neq 7.5x_2) \vee (x_2 = 5)$.

[1] Unfortunately, it seems like Hill will be in need of juridicial assistance.

Throughout this paper, we assume all sets of DLRs to be finite. The definition of satisfiability for DLRs is straightforward.

Definition 3 Let $X = \{x_1, \ldots, x_n\}$ be a set of real-valued variables and let $R = \{R_1, \ldots, R_k\}$ be a set of DLRs over X. We say that R is *satisfiable* iff there exists an assignment of real values to the variables in X that makes at least one member of each R_i, $1 \leq i \leq k$, true.

It is important to note that we only consider assignments of *real* values, thus assuming that time is linear, dense and unbounded. (We will see that it is sufficient to consider assignments of rational values further on.) We continue by classifying different types of DLRs.

Definition 4 Let γ be a DLR. $\mathcal{C}(\gamma)$ denotes the convex relations in γ and $\mathcal{NC}(\gamma)$ the disequational relations in γ. We say that γ is *convex* iff $|\mathcal{NC}(\gamma)| = 0$ and that γ is *disequational* iff $|\mathcal{C}(\gamma)| = 0$. If γ is convex or disequational we say that γ is *homogenous* and otherwise *heterogenous*. Furthermore, if $|\mathcal{C}(\gamma)| \leq 1$ then γ is *Horn*. We extend these definitions to sets of relations in the obvious way. For example, if Γ is a set of DLRs and all $\gamma \in \Gamma$ are Horn, then Γ is Horn.

This classification provides the basis for the forthcoming proofs. One detail to note is that if a Horn DLR is convex then it is a unit clause, *i.e.* a disjunction with only one member.

For Horn DLRs, we restrict ourselves only to use $\leq$ and $\neq$ in the relations. This is no loss of generality since we can express all the other relations in terms of these two. For example, a DLR of the form $x < y \vee D$ can be replaced by the disjunctions $\{x \leq y \vee D, x \neq y \vee D\}$. Observe that the resulting set of disjunctions can contain at most twice as many disjunctions as the original one. Hence, this is a polynomial time transformation. (Note, however, that this does not hold for general DLRs.)

Definition 5 Let A be a satisfiable set of DLRs and let γ be a DLR. We say that γ *blocks* A iff for every $d \in \mathcal{NC}(\gamma)$, $A \cup \{d\}$ is not satisfiable.

Observe that if $A \cup \{\gamma\}$ is satisfiable and γ blocks A then there must exist a relation $\delta \in \mathcal{C}(\gamma)$ such that $A \cup \{\delta\}$ is satisfiable. This observation will be of great importance in forthcoming sections.

Linear Programming

Our method for deciding satisfiability of Horn DLRs will be based on linear programming techniques and we will provide the basic facts needed in this section. The linear programming problem is defined as follows.

Definition 6 Let A be an arbitrary $m \times n$ matrix of rationals in finite precision and let $x = (x_1, \ldots, x_n)$ be an n-vector of variables over the real numbers. Then an instance of the *linear programming* (LP) problem is defined by: $\{\min c^{\mathrm{T}} x$ subject to $Ax \leq b\}$ where b is an m-vector of rationals and c an n-vector of rationals. The computational problem is as follows:

1. Find an assignment to the variables $x_1, \ldots, x_n$ such that the condition $Ax \leq b$ holds and $c^{\mathrm{T}} x$ is minimial subject to these conditions, or
2. Report that there is no such assignment, or
3. Report that there is no lower bound for $c^{\mathrm{T}} x$ under the conditions.

Analogously, we can define an LP problem where the objective is to maximize $c^{\mathrm{T}} x$ under the condition $Ax \leq b$. We have the following important theorem.

Theorem 7 The linear programming problem is solvable in polynomial time.

Observe that the restriction to finite precision is not a restriction in practice since computers are (almost without exception) using finite precision arithmetics. Several polynomial algorithms have been developed for solving LPs. Well-known examples are the algorithms by Khachiyan (1979) and Karmarkar (1984). Despite their theoretical value, it is not at all clear that they out-perform the simplex algorithm which is exponential in the worst case (Klee & Minty 1972). In fact, recent theoretical analyses lend support to its favourable average-case performance. (See for example (Smale 1983).) In the following, we assume all coeffecients to be rationals represented in finite precision.

Satisfiability of Horn DLRs

In this section we present a polynomial algorithm for deciding satisfiability of Horn DLRs. The algorithm can be found in Figure 1. The problem of deciding satisfiability for a set of Horn DLRs is denoted HornDLRSat. We begin by exhibiting a simple method for deciding whether a set of convex linear relations augmented with one disequation is satisfiable or not. There may be more efficient methods for checking this than the one we propose. However, throughout this paper we will stress simplicity instead of tuning efficiency.

Lemma 8 Let A be an arbitrary $m \times n$ matrix, b be an m-vector and $x = (x_1, \ldots, x_n)$ be an n-vector of variables over the real numbers. Let α, β be linear polynomial over $x_1, \ldots, x_n$. Deciding whether the system $S = \{Ax \leq b, \alpha \neq \beta\}$ is satisfiable or not is polynomial.

Proof: Let $\alpha' = \alpha - c$ and $\beta' = \beta - d$ where c and d are the constant terms in α and β, respectively. Consider the following instances of LP:

LP1= $\{\min\ \alpha' - \beta'$ subject to $Ax \leq b\}$

LP2= $\{\max\ \alpha' - \beta'$ subject to $Ax \leq b\}$

If LP1 and LP2 have no solutions then S is not satisfiable. If both LP1 and LP2 yield the same optimal value $d - c$ then S is not satisfiable since every solution to LP1 and LP2 forces α to equal β. Otherwise S is obviously satisfiable. Since we can solve the LP problem in polynomial time by Theorem 7, the lemma follows. $\square$

Before proceeding, we recapitulate some standard mathematical concepts.

Definition 9 Given two points $x, y \in R^n$, a *convex combination* of them is any point of the form $z = \lambda x + (1-\lambda) y$ where $0 \leq \lambda \leq 1$. A set $S \subseteq R^n$ is *convex* iff it contains all convex combinations of all pairs of points $x, y \in S$.

Definition 10 A *hyperplane* H in R^n is a non-empty set defined as $\{x \in R^n \,|\, a_1 x_1 + \ldots + a_n x_n = b\}$ for some $a_1, \ldots, a_n, b \in R$.

Definition 11 Let A be an arbitrary $m \times n$ matrix and b be an m-vector. The *polyhedron* defined by A and b is the set $\{x \in R^n | Ax \leq b\}$.

The connection between polyhedrons and convex sets is expressed in the following well-known fact.

Fact 12 Every non-empty polyhedron is convex.

Consequently, the convex relations in a set of Horn DLRs defines a convex set in R^n. Furthermore, we can identify the disequations with hyperplanes in R^n. These observations motivate the next lemma.

Lemma 13 Let $S \subseteq R^n$ be a convex set and let $H_1, \ldots, H_k \subseteq R^n$ be distinct hyperplanes. If $S \subseteq \bigcup_{i=1}^{k} H_i$ then there exists a j, $1 \leq j \leq k$ such that $S \subseteq H_j$.

Proof: If it is possible to drop one or more hyperplanes from H and still have a union containing S then do so. Let $H' = \{H'_1, \ldots, H'_m\}$, $m \leq k$, be the resulting minimal set of hyperplanes. Every $H'_i \in H'$ contains some point x_i of S not in any other $H'_j \in H'$. We want to prove that there is only one hyperplane in H'.

If this is not the case, consider the line segment L adjoining x_1 and x_2. (The choice of x_1 and x_2 is not important. Every choice of x_i and x_j, $1 \leq i, j \leq m$ and $i \neq j$, would do equally well.) By convexity, $L \subseteq S$. Each $H'_i \in H'$ either contains L or meets it in at most one point. But no $H'_i \in H'$ can contain L, since then it would contain both x_1 and x_2. Thus each H'_i has at most one point in common with L, and the rest of L would not be a subset of $\bigcup_{i=1}^{m} H'_i$ which contradicts that $L \subseteq S \subseteq \bigcup_{i=1}^{m} H'_i$. $\square$

We can now tie together the results and end up with a sufficient condition for satisfiability of Horn DLRs.

Lemma 14 Let Γ be a set of arbitrary Horn DLRs. Let $C \subseteq \Gamma$ be the set of convex DLRs in Γ and let $D = \{D_1, \ldots, D_k\} \subseteq \Gamma$ be the set of DLRs that are not convex. Under the condition that C is satisfiable, Γ is satisfiable if D_i does not block C for any $1 \leq i \leq k$.

Proof: Pick one disequation d_i out of every D_i such that $\{C, d_i\}$ is satisfiable. This is possible since no D_i blocks C. We show that $\Gamma' = \{C, d_1, \ldots, d_k\}$ is

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1  algorithm SAT(Γ)
2  A ← ⋃{C(γ)|γ ∈ Γ is convex}
3  if A not satisfiable then reject
4  if ∃γ ∈ Γ that blocks A and is disequational then
   reject
5  if ∃γ ∈ Γ that blocks A and is heterogenous then
   SAT((Γ − {γ}) ∪ C(γ))
6  accept
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Figure 1: Algorithm for deciding satisfiability of Horn DLRs.

satisfiable and, hence, Γ is satisfiable. Assume that $d_i = (\alpha_i \neq \beta_i)$. Define the hyperplanes $H_1, \ldots, H_k$ such that $H_i = \{x \in R^n \mid \alpha_i(x) = \beta_i(x)\}$. Since every $\{C, d_i\}$ is satisfiable, the polyhedron P defined by C (which is non-empty and hence convex by Fact 12) is not a subset of any H_i. Suppose Γ' is not satisfiable. Then $P - \bigcup_{i=1}^{k} H_i = \varnothing$ which is equivalent with $P \subseteq \bigcup_{i=1}^{k} H_i$. By Lemma 13, there exists a H_j, $1 \leq j \leq k$ such that $S \subseteq H_j$. Clearly, this contradicts our initial assumptions. □

It is important to note that the previous lemma does not give a necessary condition for satisfiability of Horn DLRs. We claim that the algorithm in Figure 1 correctly solves HornDLRSat in polynomial time. To show this, we need an auxiliary lemma which is a formal version of an observation made in the second section of this paper.

Lemma 15 Let Γ be a set of Horn DLRs and let $C \subseteq \Gamma$ be the set of convex DLRs in Γ. If there exists a heterogenous DLR $\gamma \in \Gamma$ such that γ blocks C, then Γ is satisfiable iff $(\Gamma - \{\gamma\}) \cup \mathcal{C}(\gamma)$ is satisfiable.

Proof: *if:* Trivial.
only-if: If Γ is satisfiable, then γ has to be satisfiable. Since γ blocks C, $\mathcal{C}(\gamma)$ must be satisfied in any solution of Γ. □

We can now prove the soundness and completeness of SAT.

Lemma 16 Let Γ be a set of Horn DLRs. If SAT(Γ) accepts then Γ is satisfiable.

Proof: Induction over n, the number of heterogenous DLRs in Γ.
Basis step: If $n = 0$ and SAT(Γ) accepts then the formulae in A are satisfiable and there does not exist any $\gamma \in \Gamma$ that blocks A. Consequently, Γ is satisfiable by Lemma 14.
Induction hypothesis: Assume the claim holds for $n = k$, $k \geq 0$.
Induction step: Γ contains $k+1$ heterogenous DLRs. If SAT accepts in line 5 then $(\Gamma - \{\gamma\}) \cup \mathcal{C}(\gamma)$, which contains k heterogenous DLRs, is satisfiable by the induction hypothesis. By Lemma 15, this is equivalent with Γ being satisfiable. If SAT accepts in line 6 then there does not exist any disequational or heterogenous $\gamma \in \Gamma$ which blocks A. By Lemma 14, this means that Γ is satisfiable. □

Before proving the completeness of SAT we need the following lemma.

Lemma 17 Let Γ be a set of Horn DLRs. Let $C \subseteq \Gamma$ be the set of convex DLRs in Γ. If there exists a disequational DLR $\gamma \in \Gamma$ that blocks C then Γ is not satisfiable.

Proof: In any solution to Γ, the relations in $C \cup \{\gamma\}$ must be satisfied. Since γ is disequational and blocks C this is not possible and the lemma follows. □

Lemma 18 Let Γ be a set of Horn DLRs. If SAT(Γ) rejects then Γ is not satisfiable.

Proof: Induction over n, the number of heterogenous DLRs in Γ.
Basis step: If $n = 0$ then SAT can reject in lines 3 and 4. If SAT rejects in line 3 then, trivially, Γ is not satisfiable. If SAT rejects in line 4 then there exists a disequational $\gamma \in \Gamma$ that blocks A. Hence, Γ is not satisfiable by Lemma 17.
Induction hypothesis: Assume the claim holds for $n = k$, $k \geq 0$.
Induction step: Γ contains $k+1$ heterogenous DLRs. If SAT rejects in line 3 then Γ is not satisfiable. If SAT rejects in line 4 then Γ is not satisfiable by Lemma 17. If SAT rejects in line 5 then $(\Gamma - \{\gamma\}) \cup \mathcal{C}(\gamma)$, which contains k heterogenous DLRs, is not satisfiable by the induction hypothesis. By Lemma 15, this is equivalent with Γ not being satisfiable. □

Finally, we can show that SAT is a polynomial-time algorithm and, thus, show that HornDLRSat is polynomial.

Theorem 19 HornDLRSat is polynomial.

Proof: By Lemmata 16 and 18, it is sufficient to show that SAT is polynomial. The number of recursive calls is bounded by the number of heterogenous DLRs in the given input. By Lemma 8, we can in polynomial time decide whether a linear inequality system with one disequation is satisfiable. Since we need only check a polynomial number of such systems in each recursion, the theorem follows. □

We conclude this section with a discussion about whether HornDLRSat can be effeciently solved on parallel computers. The complexity class NC consists of the problems that can be solved with a polynomial number of processors in polylogarithmic time and it is often argued that NC captures our intuitive notion of problems satisfactorily solved by parallel computers. Recall that the satisfiability problem for propositional Horn clauses (HornSat) is P-complete under log-space reductions (Greenlaw, Hoover, & Russo 1993).

Clearly, it is trivial to reduce HORNSAT to HORNDLRSAT in log-space. Since HORNDLRSAT is polynomial, it follows that it is P-complete as well. This implies that HORNDLRSAT is not in NC and, hence, there does not exist parallel algorithms for HORNDLRSAT that is substantially faster than ordinary sequential algorithms. (Unless NC=P which is considered very unlikely.)

Comparison

In this section, we show that Horn DLRs subsumes several other methods for temporal constraint reasoning. Let x, y be real-valued variables, c, d constants and $\mathcal{A}$ Allen's algebra (Allen 1983) in the definitions below.

Definition 20 (Nebel & Bürckert 1995) An *ORD clause* is a disjunction of relations of the form xry where $r \in \{\leq, =, \neq\}$. The *ORD-Horn* subclass $\mathcal{H}$ is the relations in $\mathcal{A}$ that can be written as ORD clauses containing only disjunctions with at most one relation of the form $x = y$ or $x \leq y$ and an arbitrary number of relations of the form $x \neq y$.

Note that the ORD-Horn class subsumes both the continuous endpoint algebra (Vilain, Kautz, & van Beek 1989) and the pointisable endpoint algebra (van Beek & Cohen 1990).

Definition 21 (Koubarakis 1992) Let $r \in \{\leq, \geq, \neq\}$. A *Koubarakis formula* is a formula on one of the following forms (1) $(x-y)rc$, (2) xrc or (3) a disjunction of formulae of the form $(x-y) \neq c$ or $x \neq c$.

Definition 22 (Dechter, Meiri, & Pearl 1991) A *simple temporal constraint* is a formula on the form $c \leq (x-y) \leq d$.

Simple temporal constraints are equivalent with the *simple metric constraints* (Kautz & Ladkin 1991).

Definition 23 (Meiri 1991) A *CPA/single interval* formula is a formula on one of the following forms: (1) $c\, r_1\, (x-y)\, r_2\, d$; or (2) $x\, r\, y$ where $r \in \{<, \leq, =, \neq, \geq, >\}$ and $r_1, r_2 \in \{<, \leq\}$.

Definition 24 (Gerevini, Schubert, & Schaeffer 1993) A *TG-II* formula is a formula on one of the following forms: (1) $c \leq x \leq d$, (2) $c \leq x - y \leq d$ or (3) $x\, r\, y$ where $r \in \{<, \leq, =, \neq, \geq, >\}$.

We can now state the main theorem of this section.

Theorem 25 The formalisms defined in Definitions 20 to 24 can trivially be expressed as Horn DLRs.

Note that Meiri (1991) considers two further tractable classes that cannot (in any obvious way) be transformed into Horn DLRs. The finding that the ORD-Horn algebra can be expressed as Horn DLRs is especially important in the light of the following theorem.

Theorem 26 (Nebel & Bürckert 1995) Let $\mathcal{S}$ be any subclass of $\mathcal{A}$ that contains all basic relations. Then either

1. $\mathcal{S} \subseteq \mathcal{H}$ and the satisfiability problem for $\mathcal{S}$ is polynomial, or
2. Satisfiability for $\mathcal{S}$ is NP-complete.

By the previous theorem, we cannot expect to find tractable classes that are able to handle all basic relations in $\mathcal{A}$ and, at the same time, are able to handle any single relation that cannot be expressed as a Horn DLR. In other words, the qualitative fragment of HORNDLRSAT inherits the maximality of the ORD-Horn algebra.

Discussion

Several researchers in the field of temporal constraint reasoning have expressed a feeling that their proposed methods should be extended so they can express relations between more than two time points. As a first example, in (Dechter, Meiri, & Pearl 1991) one can read "The natural extension of this work is to explore TCSPs with higher-order expressions (e.g. "John drives to work at least 30 minutes more than Fred does"; $X_2 - X_1 + 30 \leq X_4 - X_3$)..." Even though they do not define the exact meaning of "higher-order expressions" we can notice that their example is a simple Horn DLR. Something similar can be found in (Koubarakis 1992) who wants to express "the duration of interval I exceeds the duration of interval J". Once again, this can easily be expressed as a Horn DLR. These claims seem to indicate that the use of Horn DLRs is a significant contribution to temporal reasoning.

We have shown that the satisfiability problem for Horn DLRs can be carried out in polynomial time. However, the method builds on solving linear programs and it is well-known that such calculations can be computationally heavy. It is important to remember the reasons for introducing Horn DLRs. The main reason was not to provide an extremely efficient method, but to find a method unifying most of the other tractable classes reported. It is fairly obvious that the proposed method cannot outperform highly specialized algorithms for severely restricted classes. It should be likewise obvious that the specialized methods cannot compete with Horn DLRs in terms of expressivity. We are, as always in tractable reasoning, facing the tradeoff between expressivity and computational complexity. We believe, though, that the complexity of deciding satisfiability can be drastically improved by devising better algorithms than SAT. The algorithm SAT is constructed in a way that facilitates its correctness proofs and it is not optimized with respect to execution time in any way. The question whether improved versions can compete with algorithms such as TIMEGRAPH II or not remains open.

Throughout this paper we have assumed that time is linear, dense and unbounded but this may not be the case in real applications. For example, in a sampled system we cannot assume time to be dense. One question to answer in the future is what the effects of changing the assumptions of time are. Switching to discrete time will probably make reasoning computationally harder. There are some positive results concerning discrete time, however. Meiri (1991) presents a class of temporal constraint reasoning problems where integer time satisfiability is polynomial.

Conclusion

We have introduced the Horn DLR as a means for temporal constraint reasoning. We have proven that deciding satisfiability of sets of Horn DLRs is polynomial by exhibiting an algorithm based upon linear programming. Furthermore, we have shown that several other approaches to tractable temporal constraint reasoning can be encoded as Horn DLRs, including the ORD-Horn algebra and most methods for purely quantitative reasoning.

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