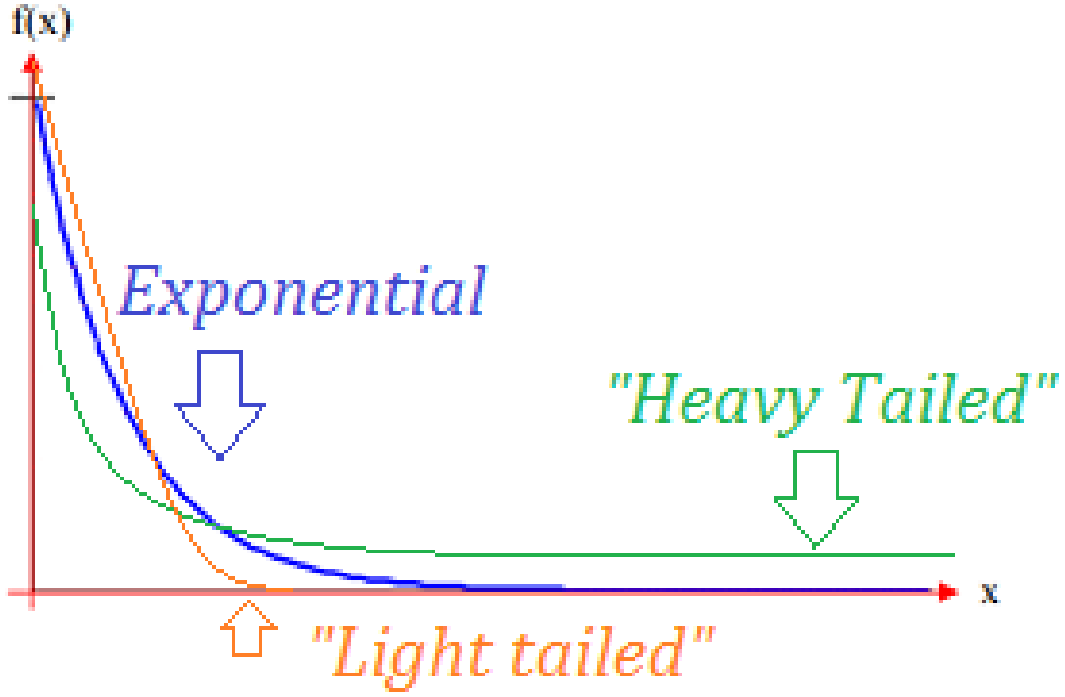


Power-laws, heavy tails, and rich-gets richer (things often observed in large-scale systems such as the internet …)

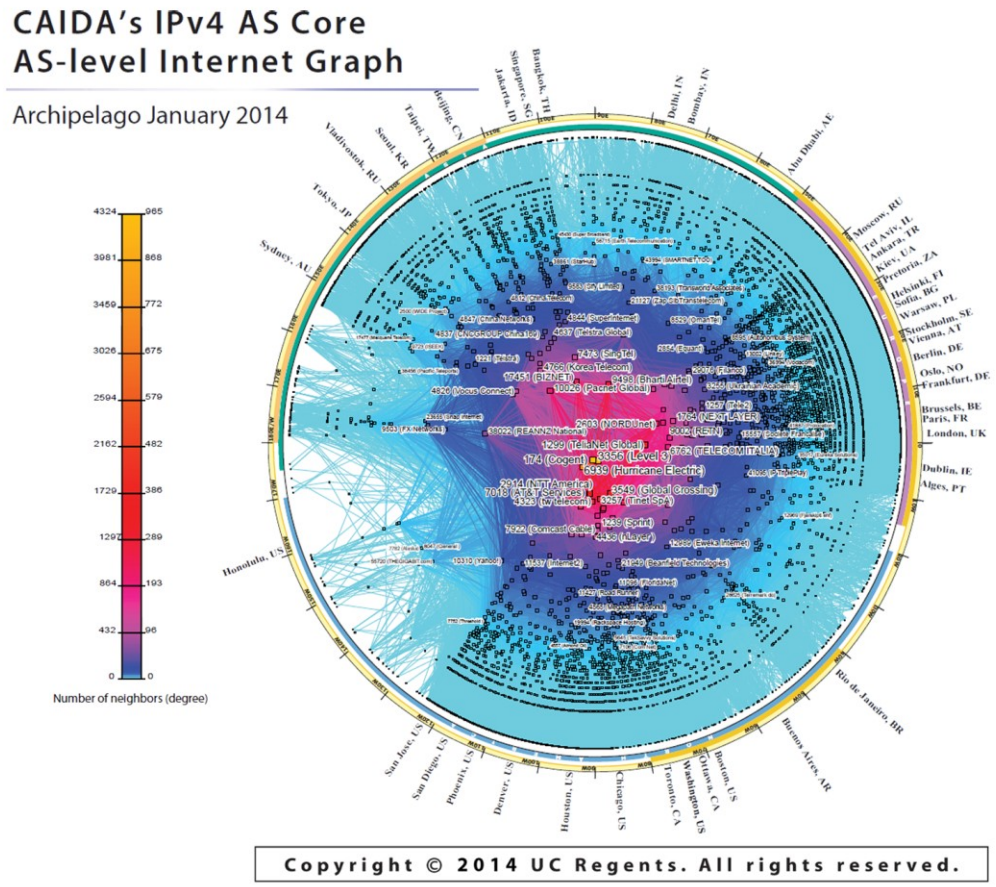
Slides by Niklas Carlsson, last revised 2021

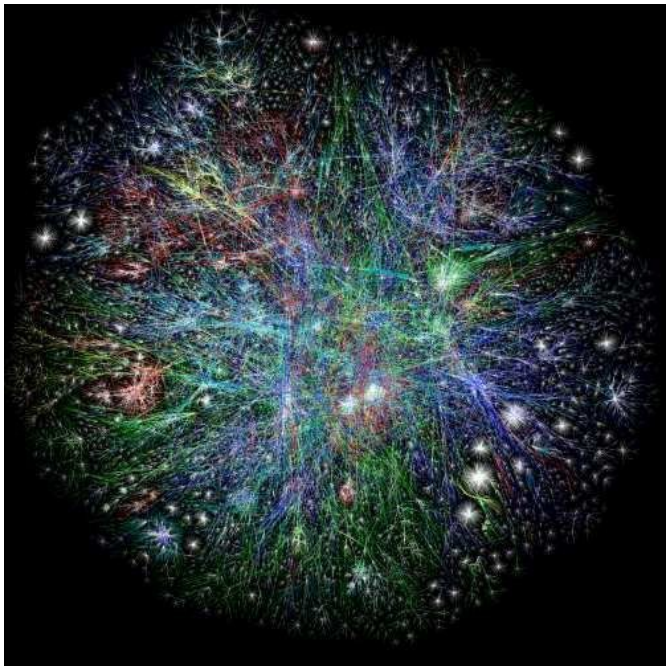
…

Things we often see in LARGE systems

- Power laws, heavy tails, and skewed distributions in general
- Preferential attachment ("Rich gets richer")

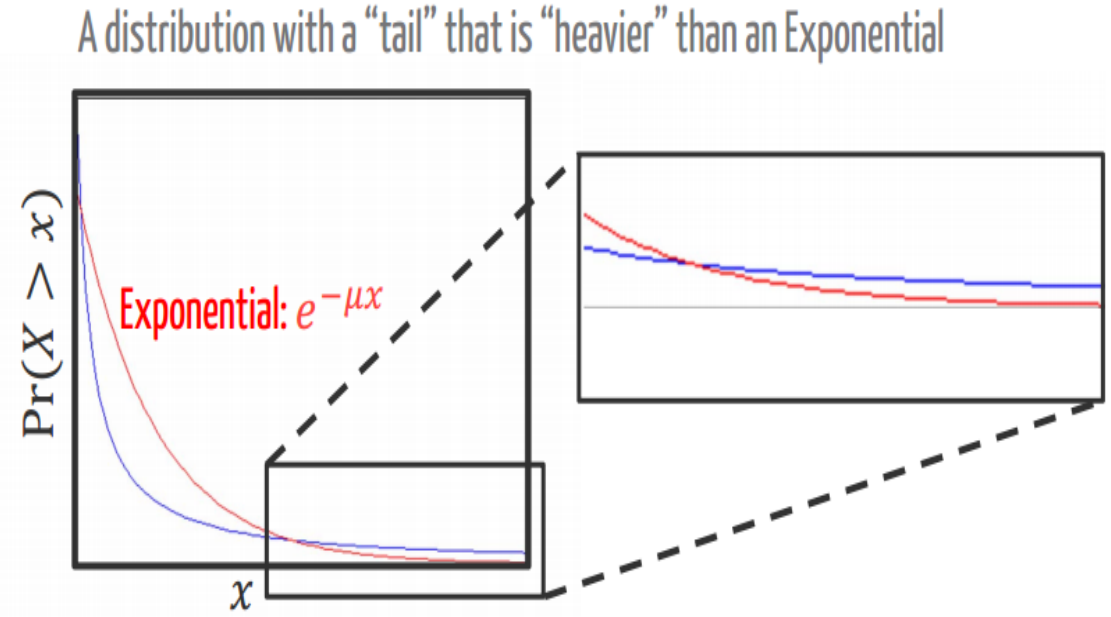
First, example from last lecture





- Examples questions one may ask:
  - What degree distribution does this graph have?
  - And what implications does that have?

Heavy-tail distributions …



- "A probability distribution is said to have a heavy tail if the tail is not exponentially bounded"
  - E.g., paper and references therein: "A Tale of the Tails: Power-laws in Internet Measurements", IEEE Network, Mahanti et al., 2013
- Power-law, Pareto, Zipf (in some sense the same)
- … and then there are many other "heavy tail" distributions, variations and generalizations, including distributions such as log-normal, various generalized Zipf/Pareto distributions, etc.

Examples of power laws

a. Word frequency: Estoup.
b. Citations of scientific papers: Price.
c. Web hits: Adamic and Huberman
d. Copies of books sold.
e. Diameter of moon craters: Neukum & Ivanov.
f. Intensity of solar flares: Lu and Hamilton.
g. Intensity of wars: Small and Singer.
h. Wealth of the richest people.
i. Frequencies of family names: e.g. US & Japan not Korea.
j. Populations of cities.

… AND many many more …

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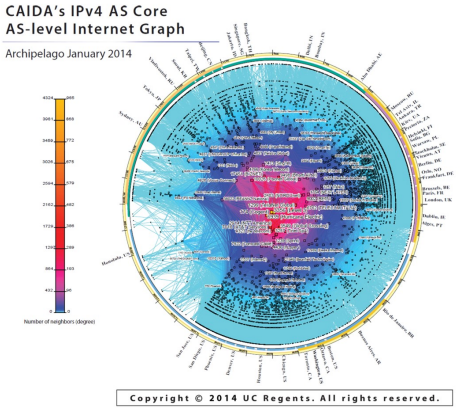
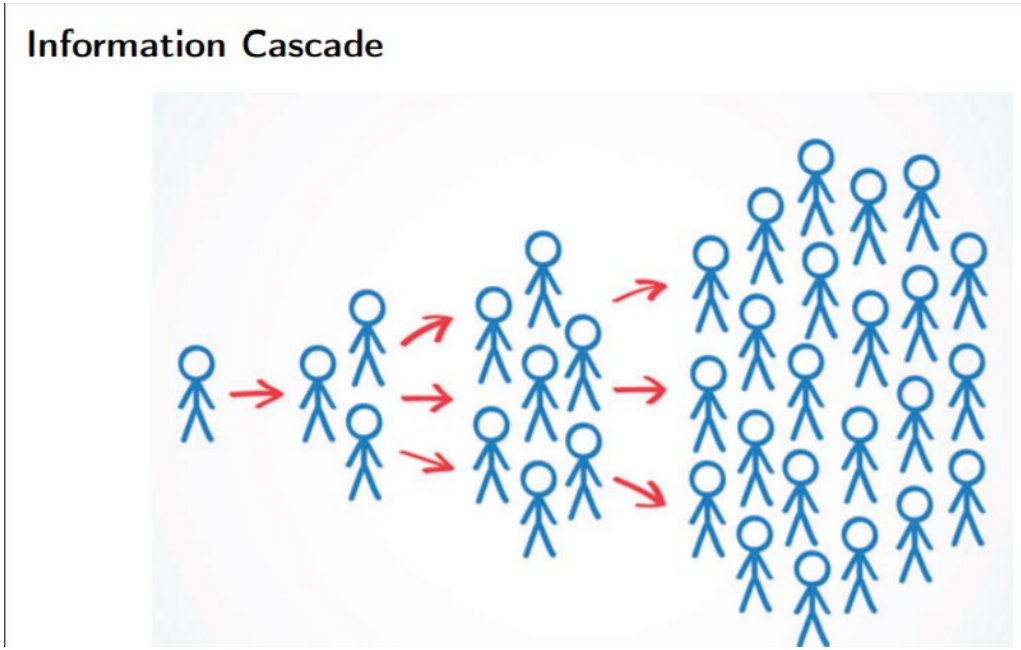
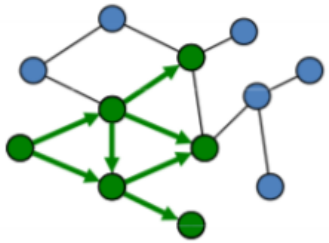


Or an even more timely example





(a) Social Network

(b) Cascade

Cascade $\Longleftrightarrow$ Propagation Graph

Cascade size $\Longleftrightarrow$ Rumour Popularity

A.Pacuk et al., Why Do Cascade Sizes Follow a Power-Law?, Proc. WWW 2017.

- The size of information cascaded, spread of fake news, and **virus reach** for that matter ...

File popularity distribution and "heavy" tails

- Example slides with YouTube popularity
  - but web object popularity, file size distributions, number of friends in social networks, etc. often see similar "heavy tail" distributions ...
  - This list can be made very-very long, and include things such as the frequency words are used, the size of cities, the size of earthquakes, the size of bacteria cultures … and the list will go on ... and on … and on ...

Motivation

- Video dissemination (e.g., YouTube) can have wide-spread impacts on opinions, thoughts, and cultures

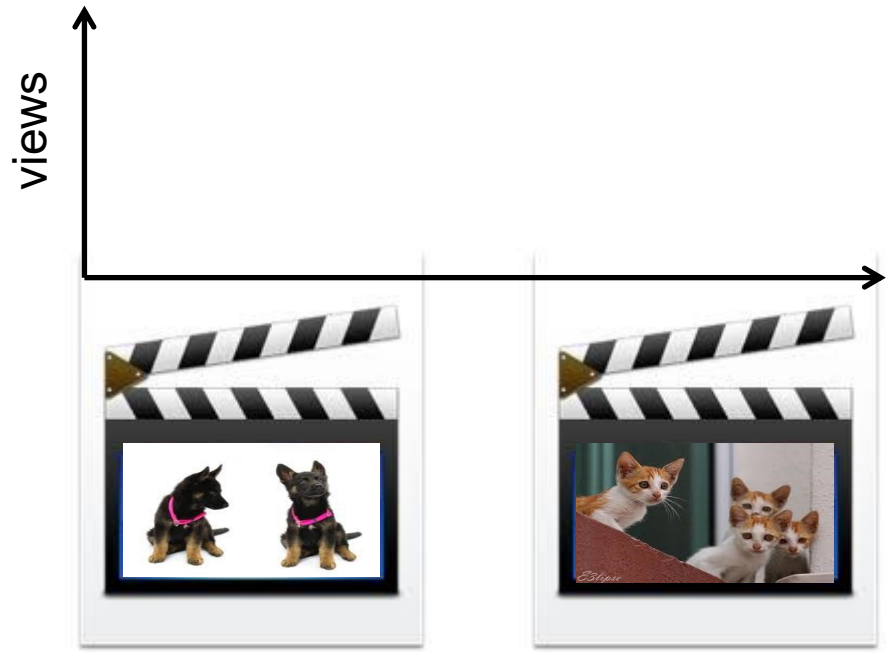
E.g., ACM KDD '12, IFIP Performance '11, ACM TWEB

Motivation

- Not all videos will reach the same popularity and have the same impact

E.g., ACM KDD '12, IFIP Performance '11, ACM TWEB

Motivation



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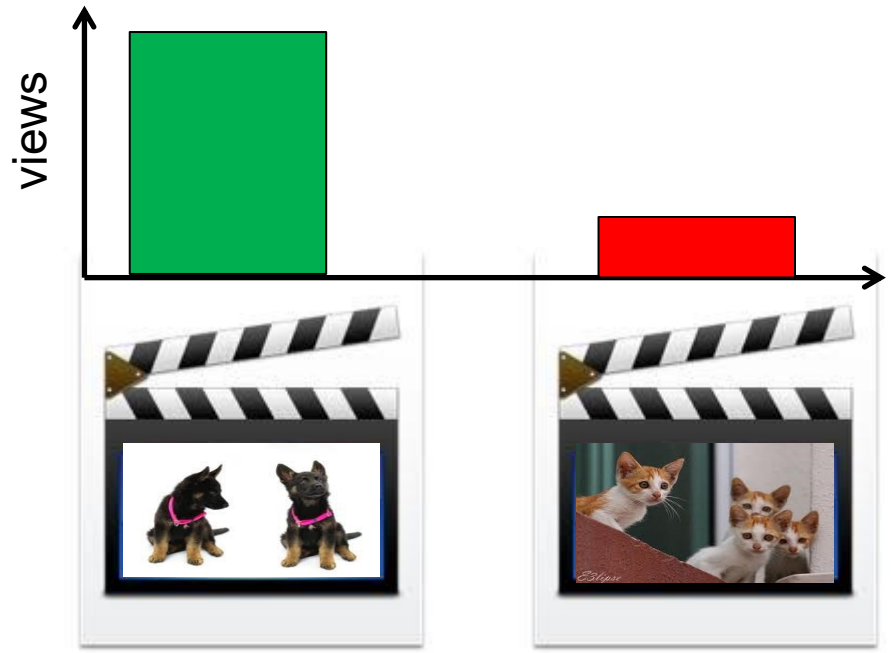

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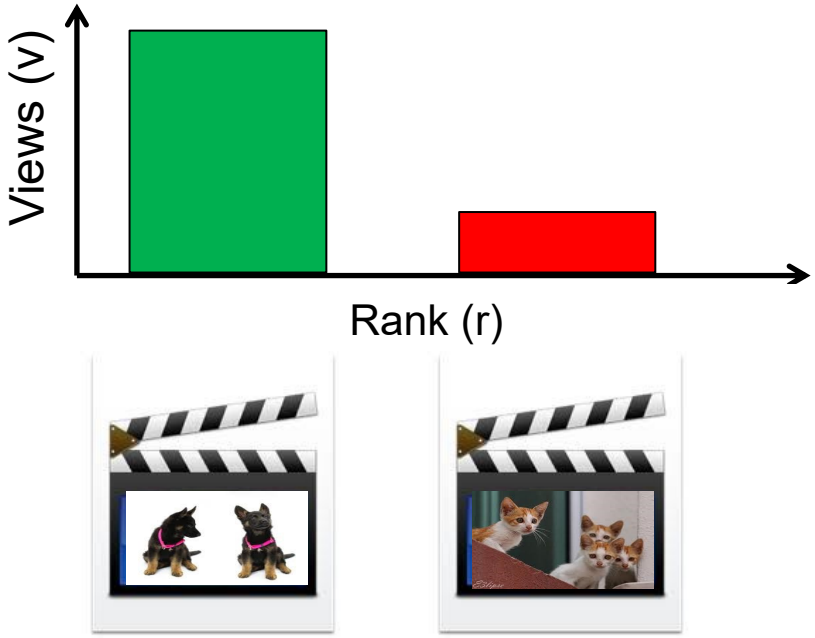
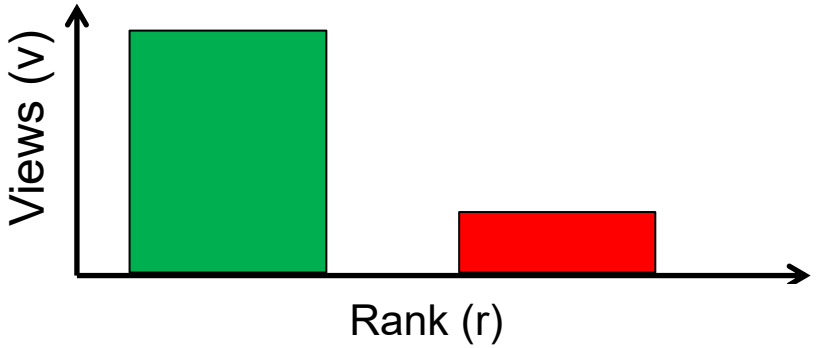
Popularity distribution



E.g., ACM KDD ‘12

Popularity distribution



E.g., ACM KDD ‘12

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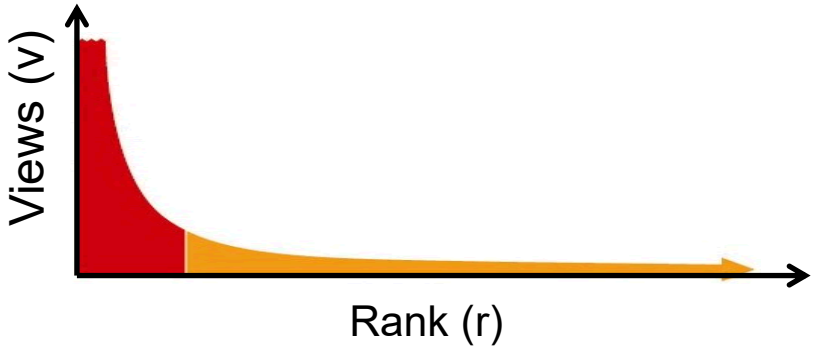
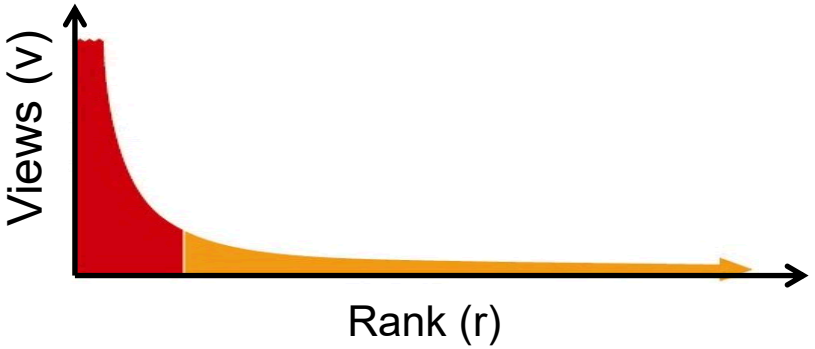
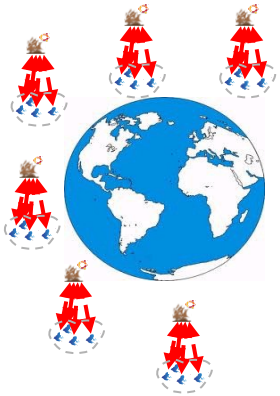


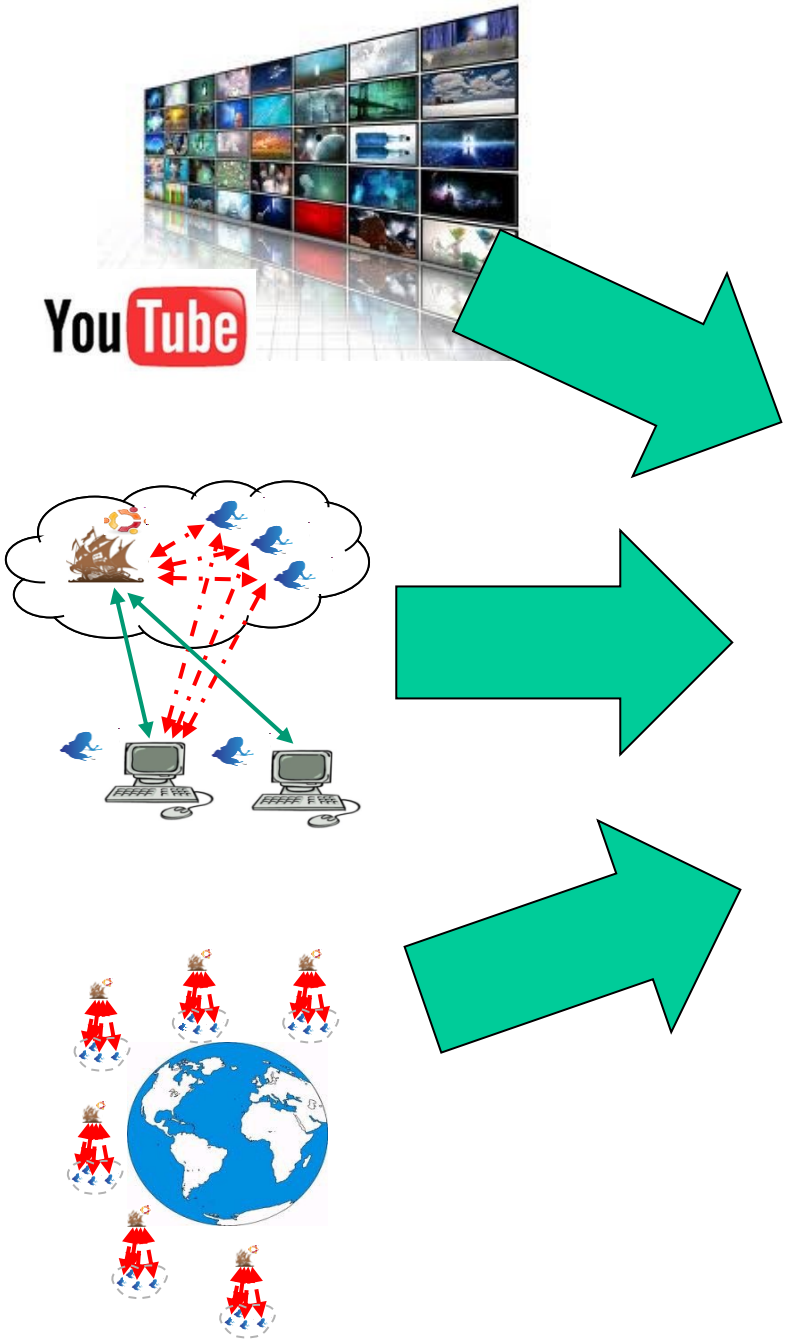
Popularity distribution

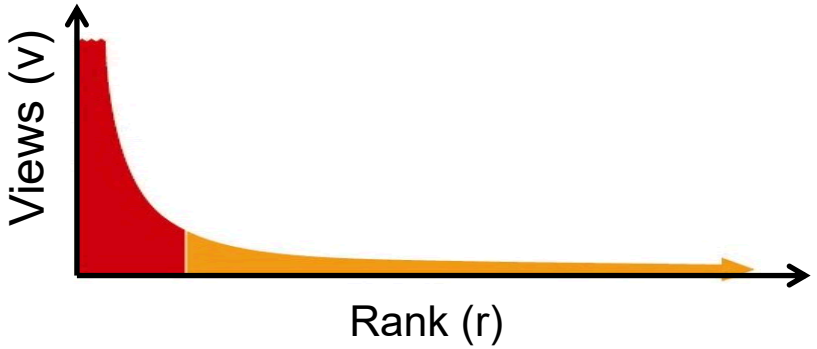




E.g., ACM KDD '12, PAM '12

Popularity distribution





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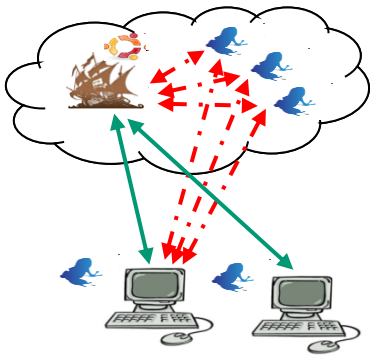
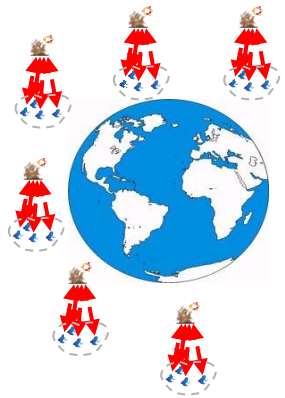
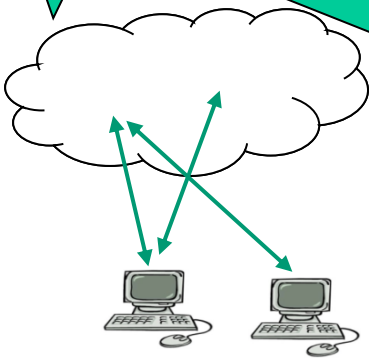
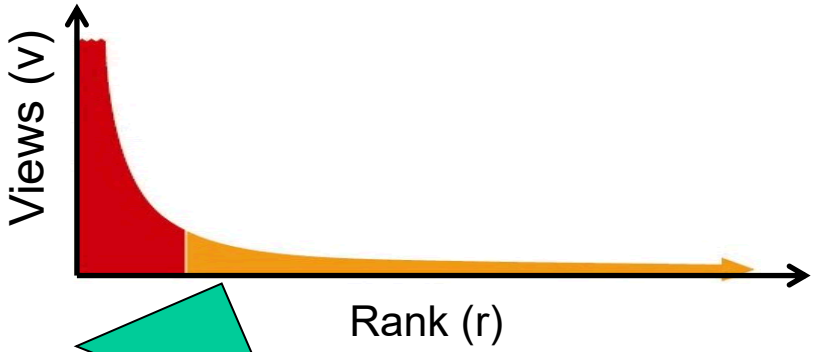






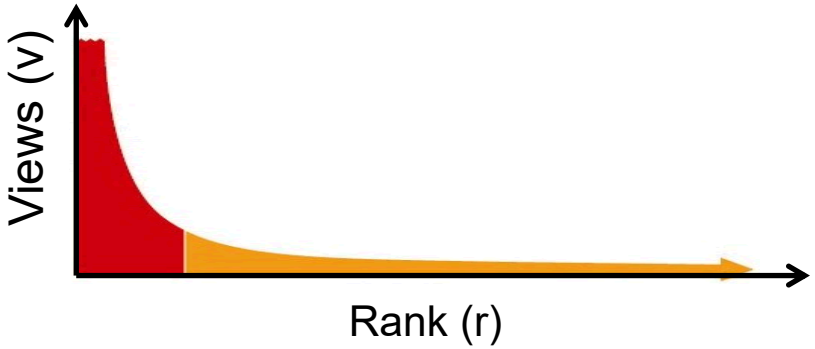


E.g., ACM KDD '12, PAM '12, ACM TWEB

Let's look at an example …

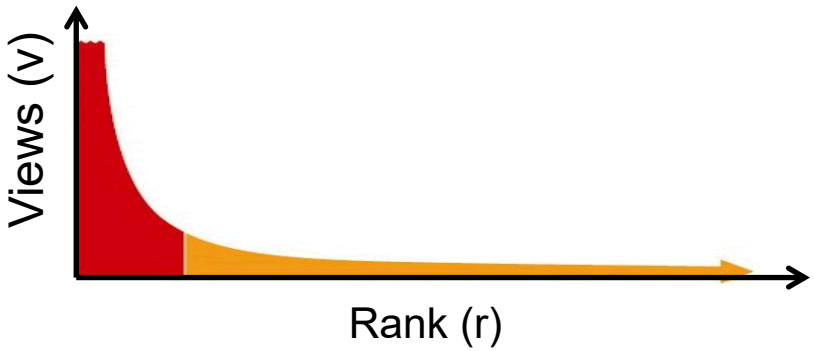
- Example 2

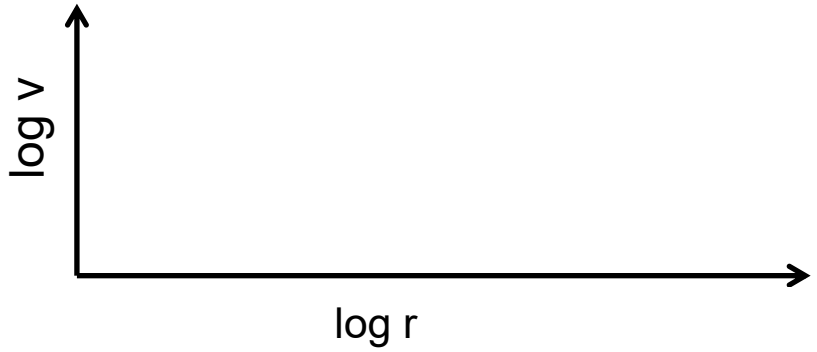
Zipf popularity...
... and long tails



$$v_r \propto r^{-\alpha}$$

Zipf popularity...
... and long tails





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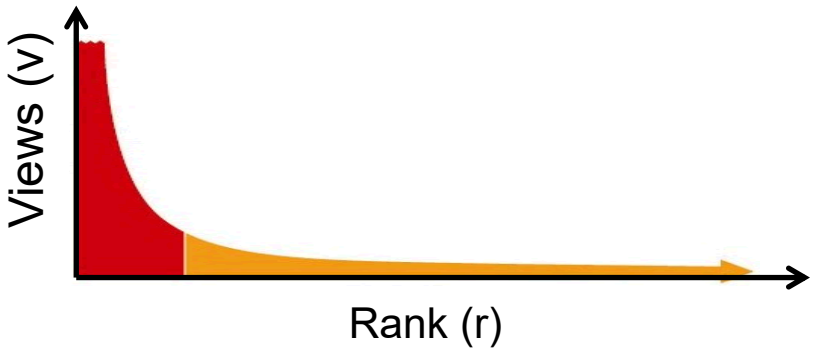

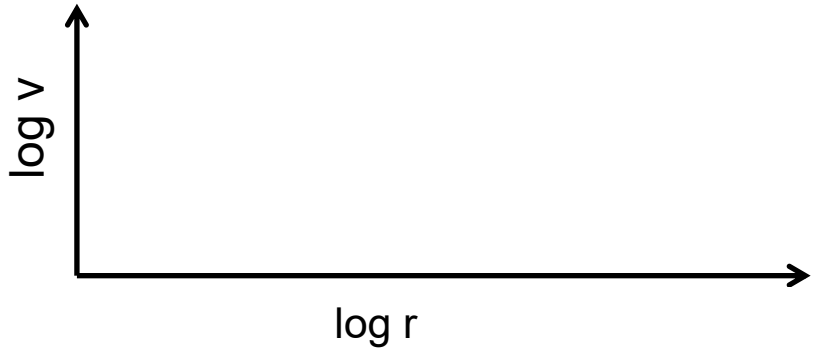

$$v_r \propto r^{-\alpha}$$

$$\log v_r = \log v_1 - \alpha \log r$$

Zipf popularity...
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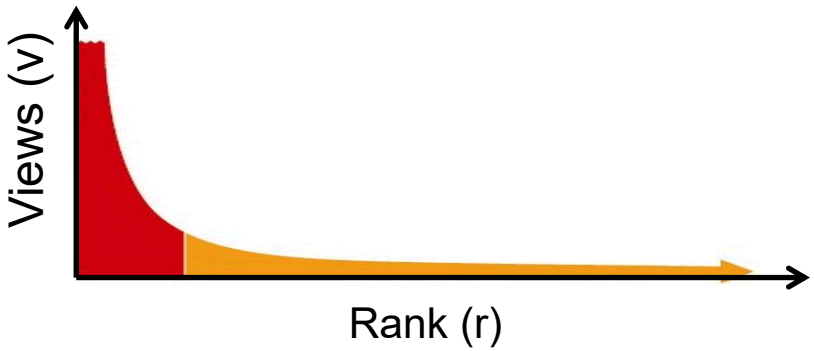

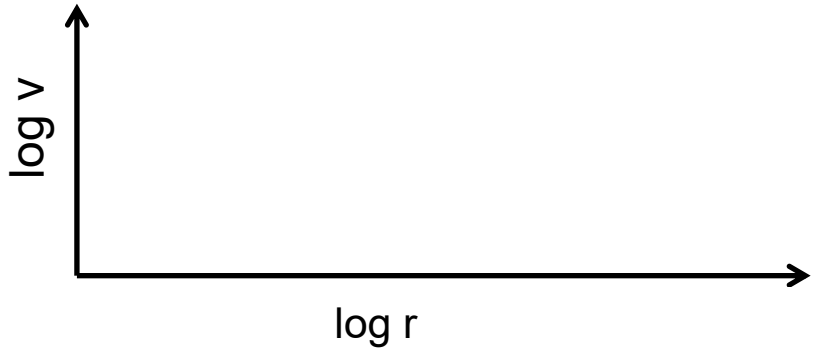

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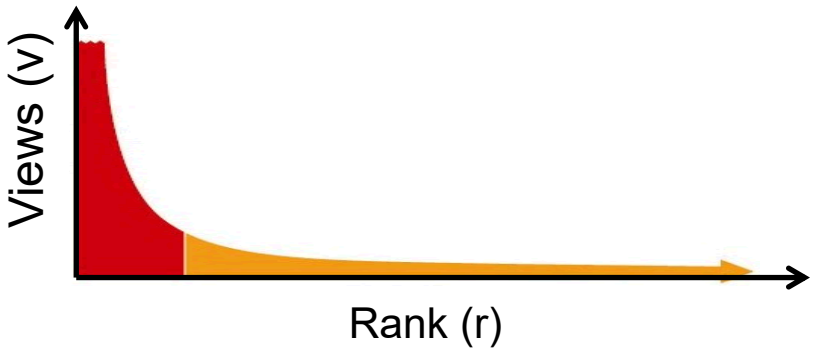

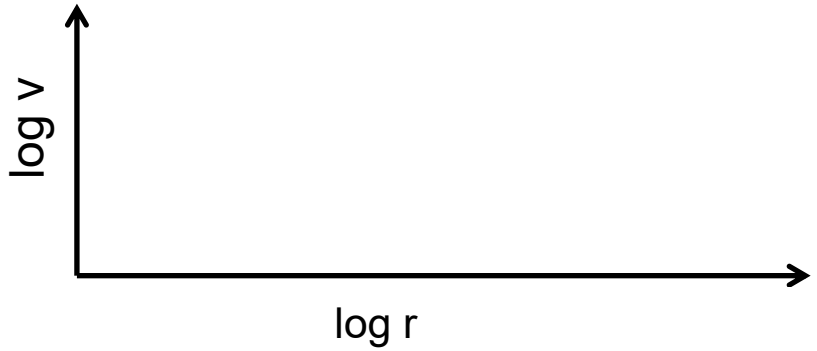

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y(x) = x_0 -- α x

Zipf popularity...
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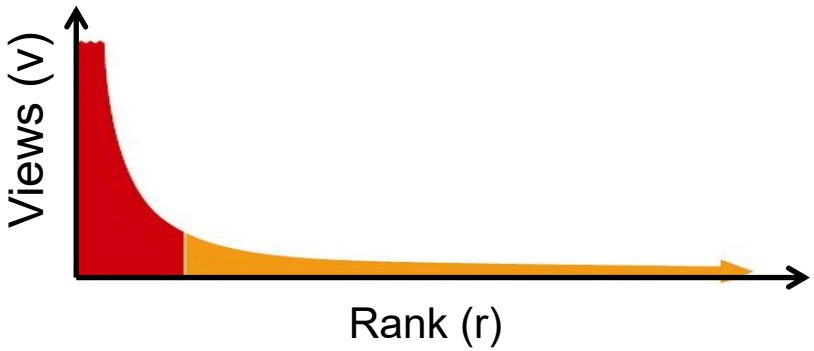

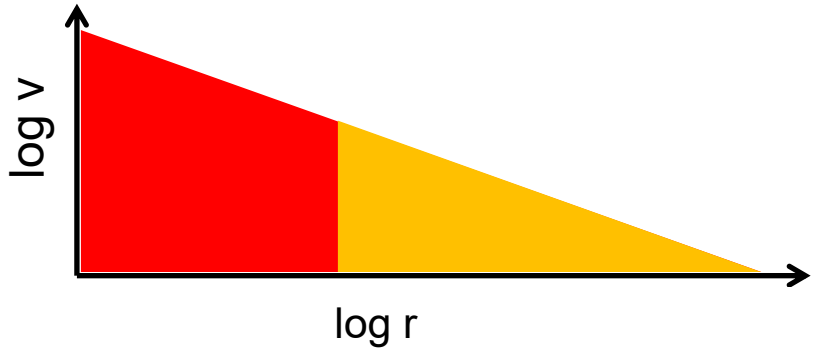

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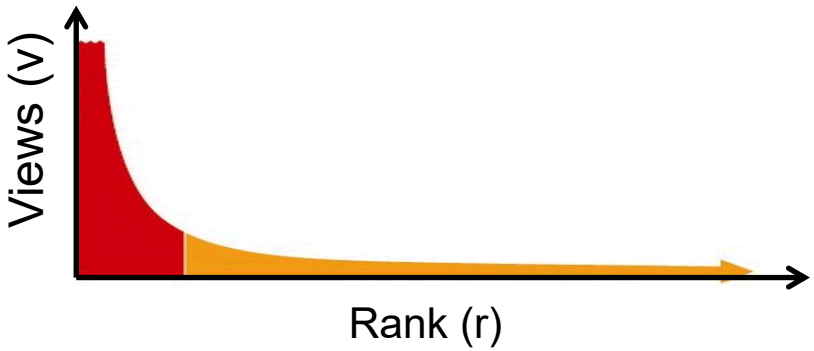

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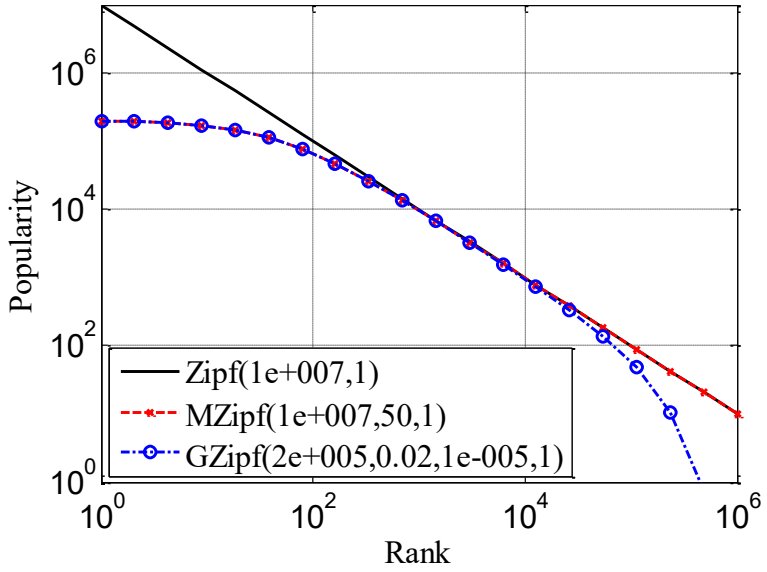

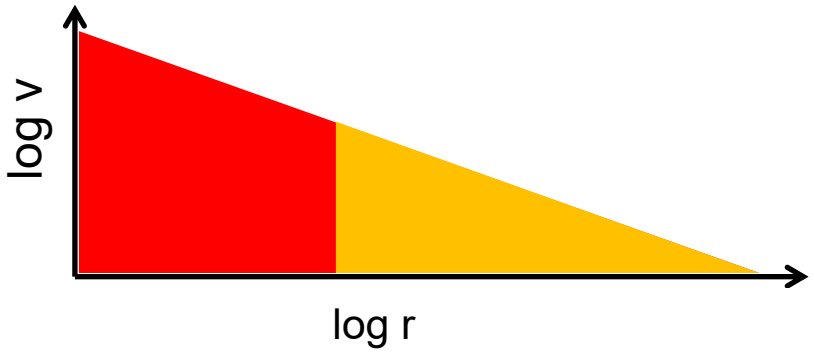

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Zipf popularity…
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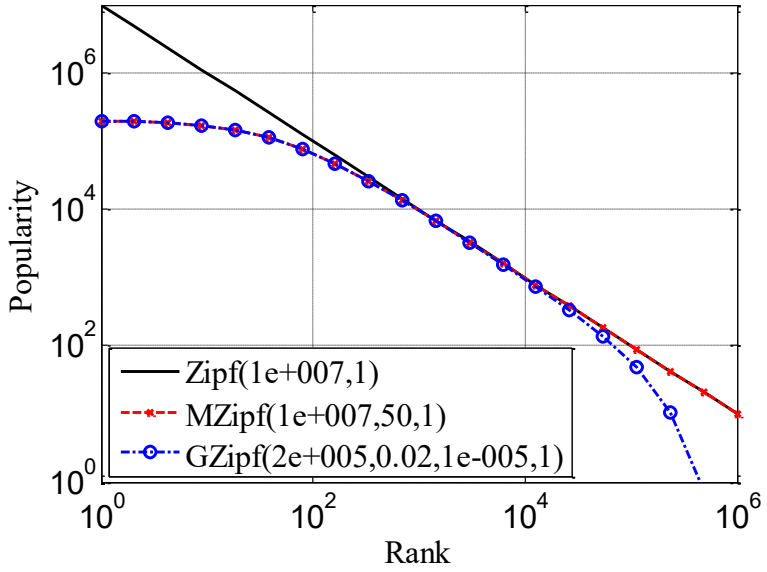
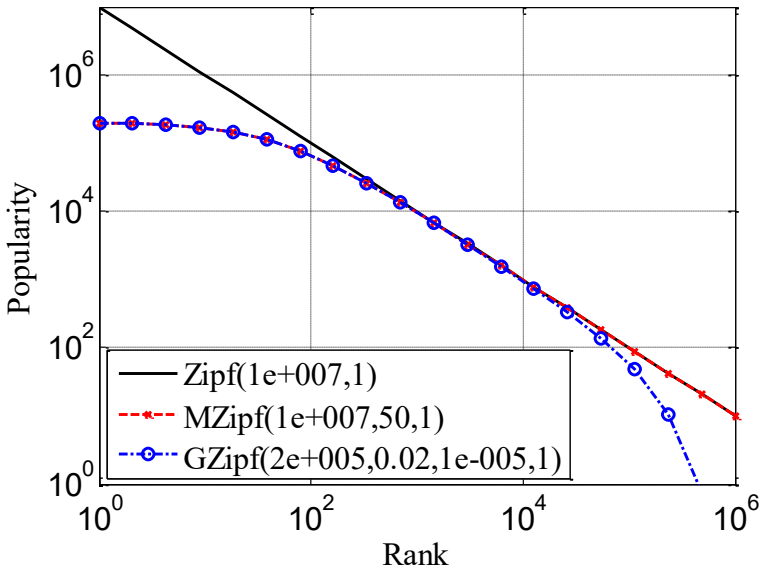
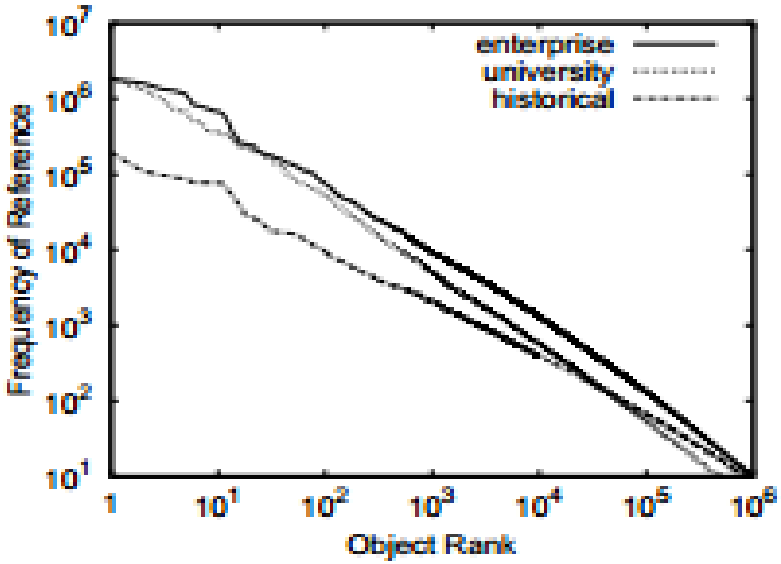


E.g., ACM TWEB, PAM ‘11
IFIP Performance ‘11, IPTPS ‘10

Zipf popularity…
… and long tails





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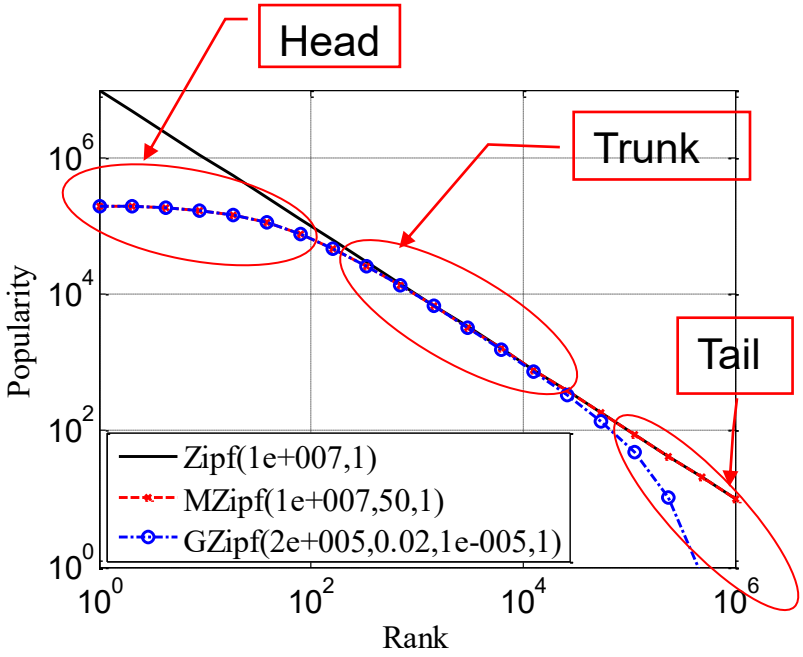
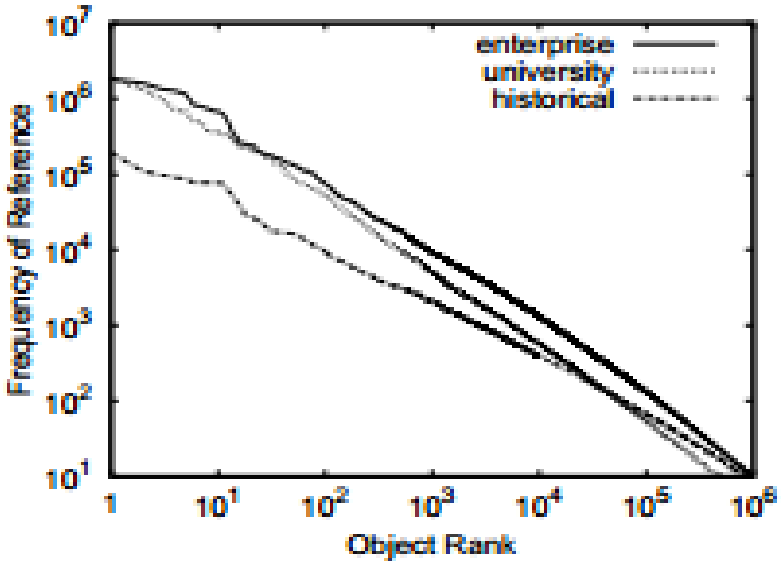


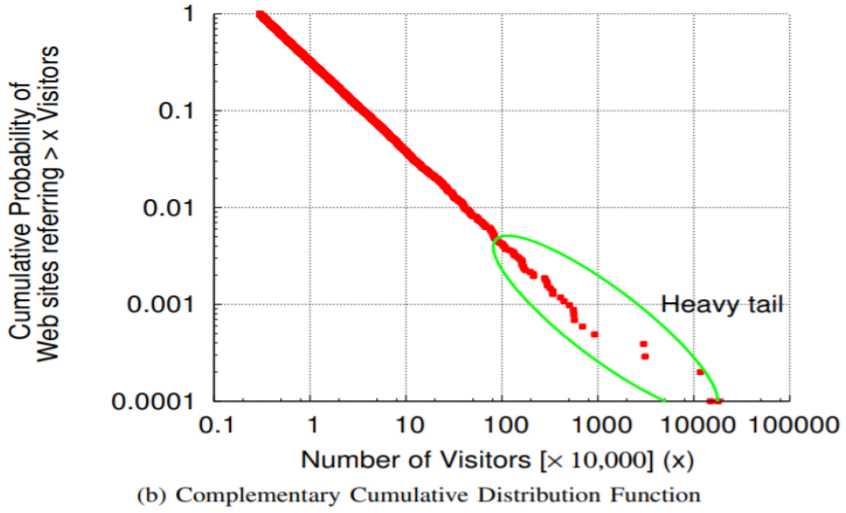


- Popularity distribution statistics
  - Across services (impact on system design)
  - Lifetime vs current
  - Over different time period (churn)
  - Different sampling methods
  - Different measurement location

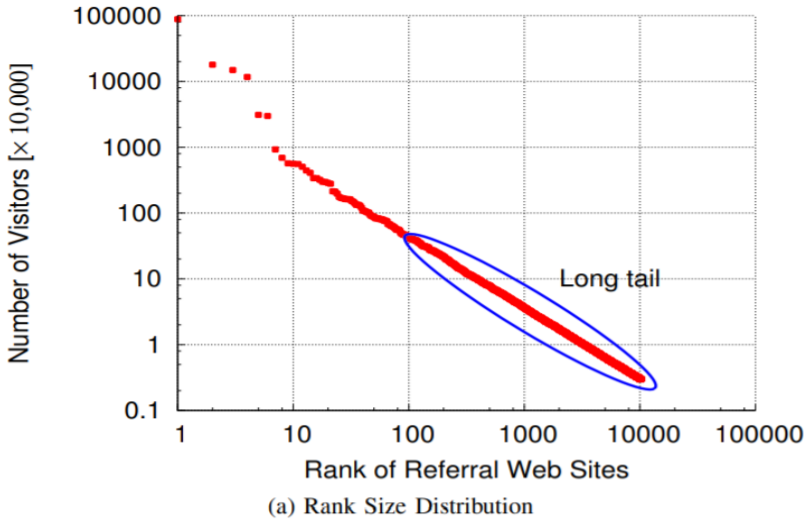
E.g., ACM TWEB, PAM '11, IFIP Performance '11, IPTPS '10

Power law, Pareto, and Zipf

- Power-law, Pareto, Zipf (in some sense the same)
  - Power-law: $f(x) \sim x^{-\eta}$ (probability of value x)
  - Pareto: $F(x) = P[X > x] = \int f(x)\,dx \propto x^{-\kappa}$ (cumulative prob.)
  - Zipf: $v_r \propto r^{-\alpha}$ (discrete representation; frequency v_r of rank r)
  - Parameters related as: $\kappa = \eta - 1 = 1/\alpha$
    - E.g., paper and references therein: "A Tale of the Tails: Power-laws in Internet Measurements", IEEE Network, Mahanti et al., 2013



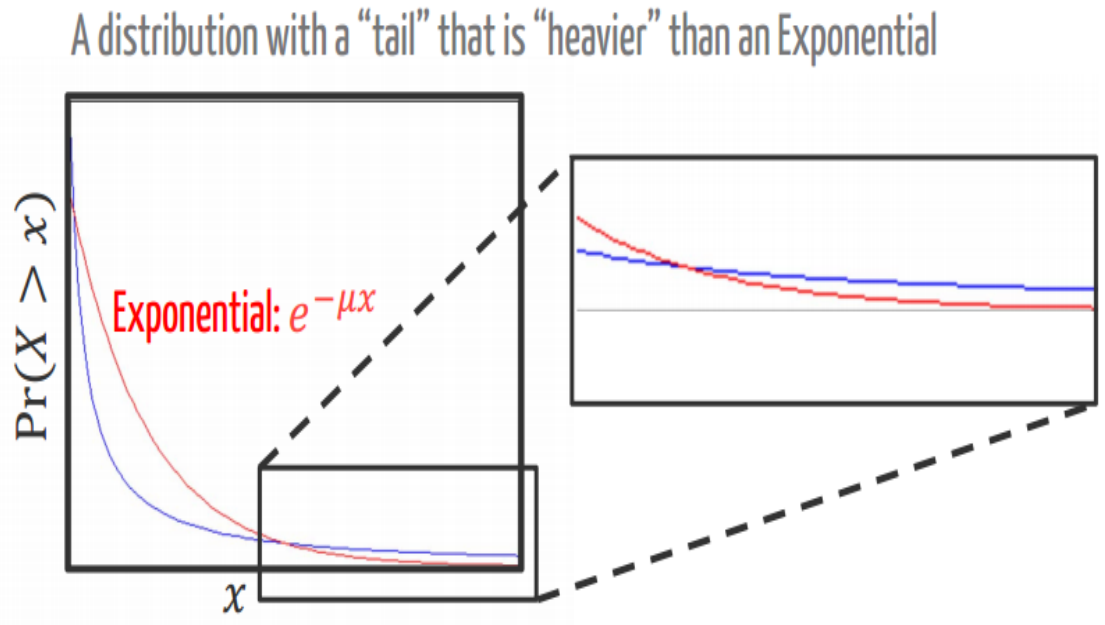
(b) Complementary Cumulative Distribution Function



(a) Rank Size Distribution

Heavy-tail distributions …

- "A probability distribution is said to have a heavy tail if the tail is not exponentially bounded"
- … and then there are many-many other "heavy tail" distributions, variations and generalizations, including distributions such as log-normal, various generalized Zipf/Pareto distributions, etc.

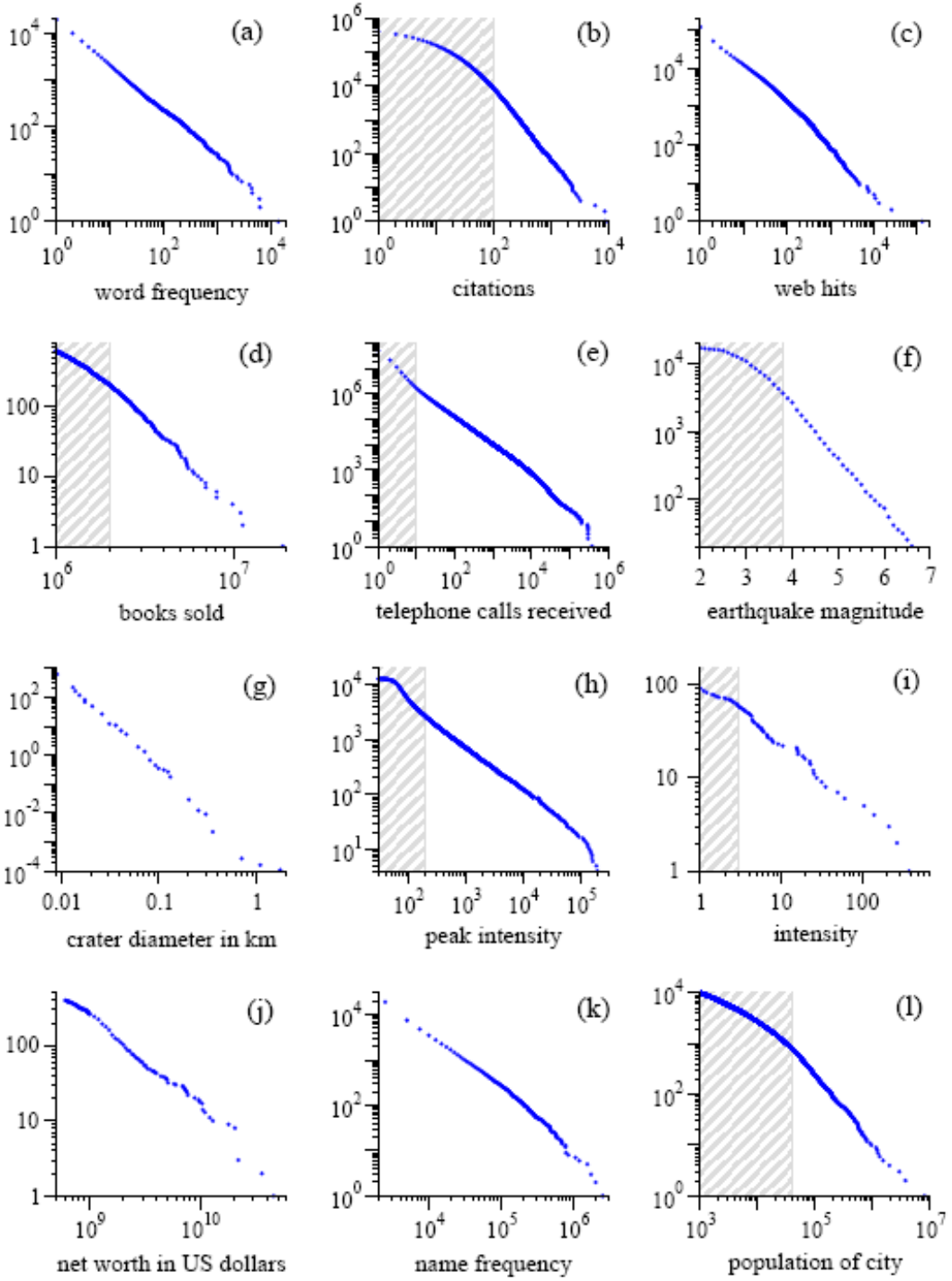


(more) Examples of power laws

a. Word frequency: Estoup.
b. Citations of scientific papers: Price.
c. Web hits: Adamic and Huberman
d. Copies of books sold.
e. Diameter of moon craters: Neukum & Ivanov.
f. Intensity of solar flares: Lu and Hamilton.
g. Intensity of wars: Small and Singer.
h. Wealth of the richest people.
i. Frequencies of family names: e.g. US & Japan not Korea.
j. Populations of cities.

… AND many many more …

The following graph is plotted using Cumulative distributions



M. E. J. Newman, "Power laws, Pareto distribution and Zipf's law", Contemporary physics (2005).

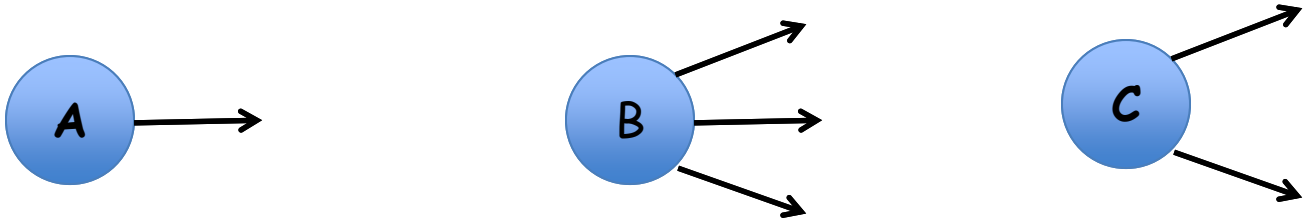
Real world data for x_{min} and α

	x_{min}	α
frequency of use of words	1	2.20
number of citations to papers	100	3.04
number of hits on web sites	1	2.40
copies of books sold in the US	2 000 000	3.51
telephone calls received	10	2.22
magnitude of earthquakes	3.8	3.04
diameter of moon craters	0.01	3.14
intensity of solar flares	200	1.83
intensity of wars	3	1.80
net worth of Americans	$600m	2.09
frequency of family names	10 000	1.94
population of US cities	40 000	2.30

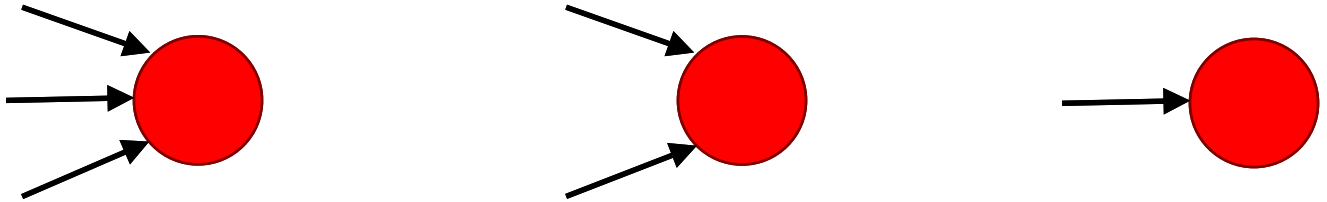
Now, consider a **social network**, the Internet, or some other network ...

Preferential Attachment (PA)

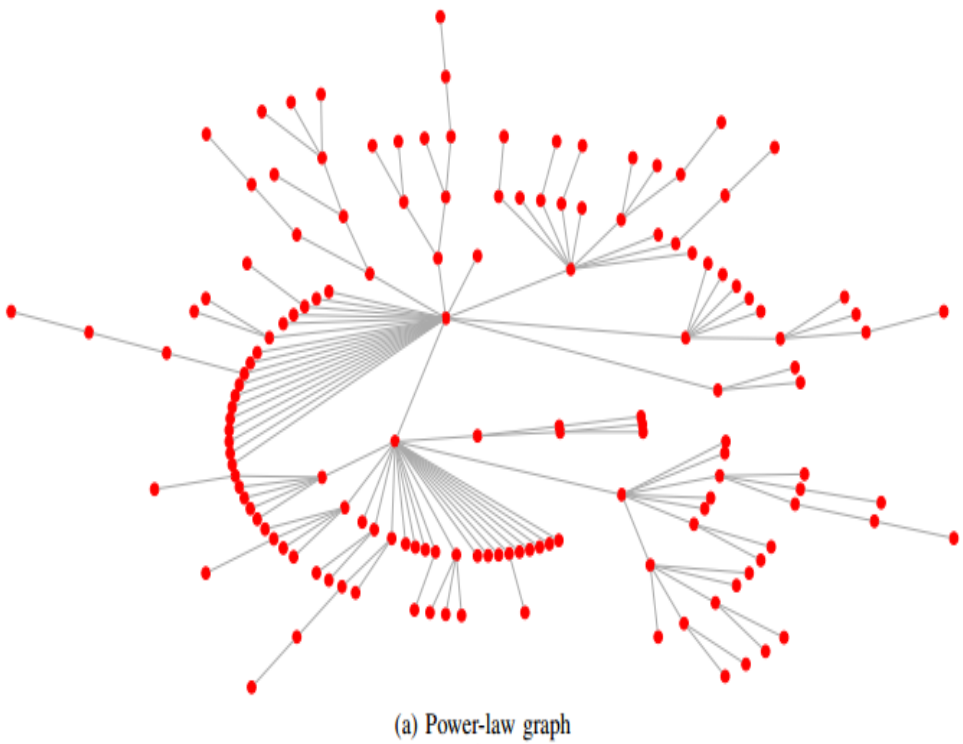
- Link probability proportional to node degree
  - p_i proportional to k_i^{α}
  - For **source node selection** (Out-degree, $\alpha = 0.8$)



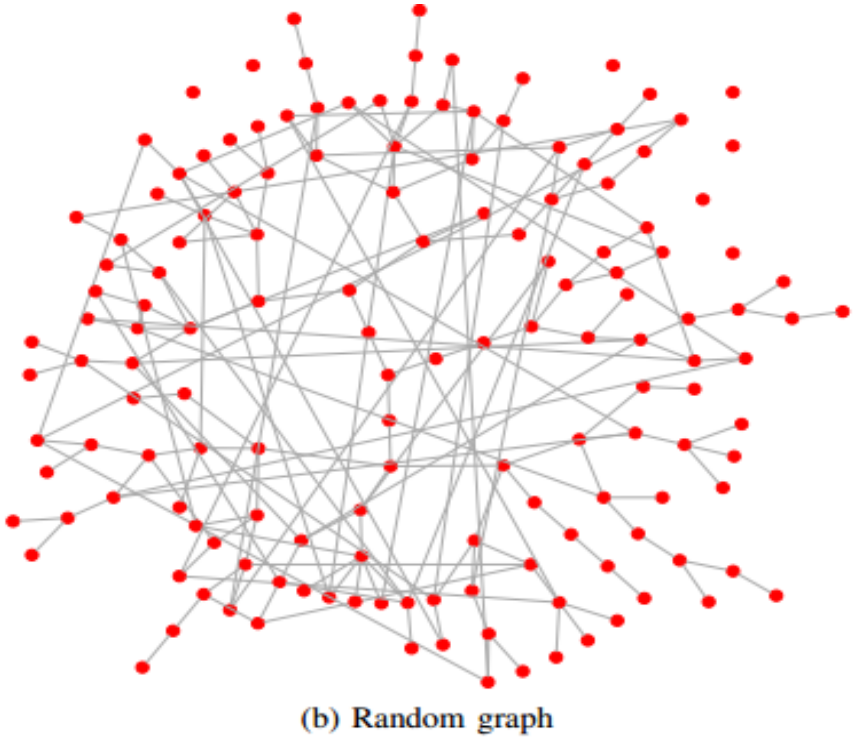
  - For **destination node selection** (In-degree, $\alpha = 0.9$)



Preferential attachment and Power law



(a) Power-law graph

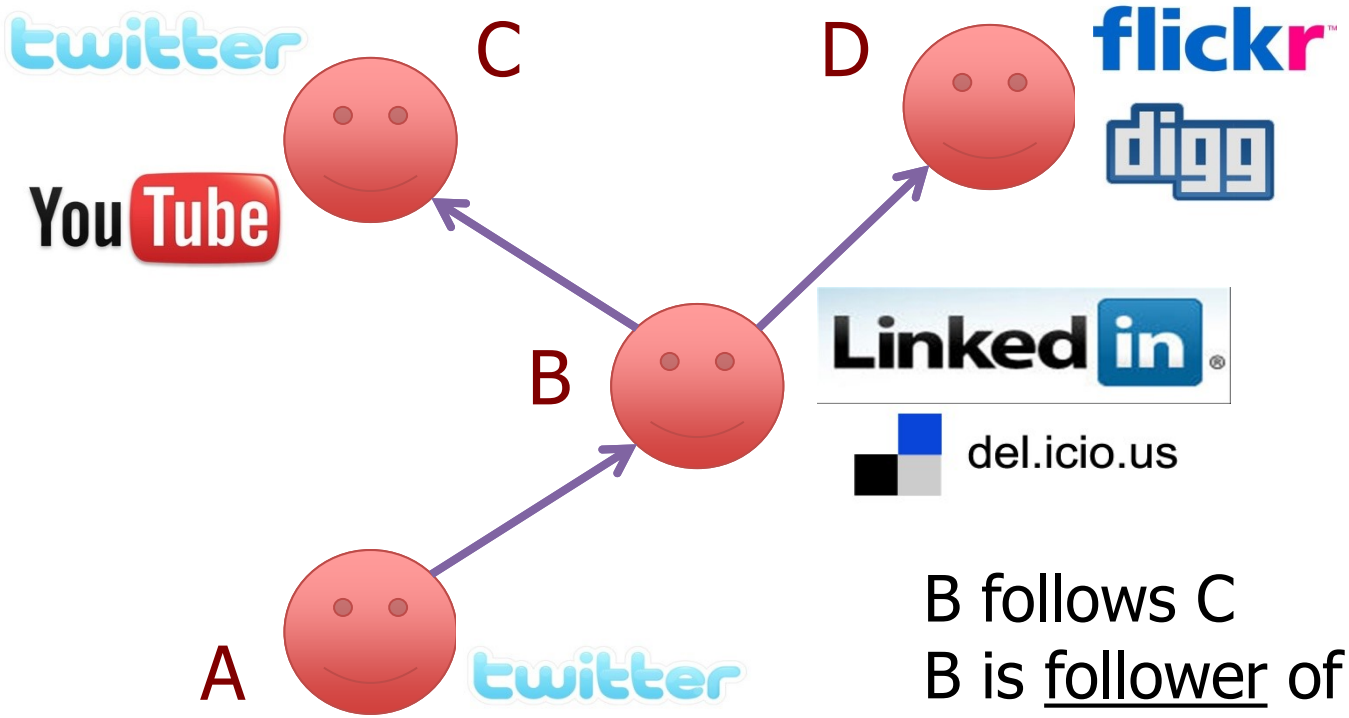


(b) Random graph

- Preferential attachment (or rich gets richer) have been shown to result in power-law graphs
- In contrast, the Erdös-Rényi random graph has an exponential node degree distribution

friendfeed

[Garg et al. IMC '09]

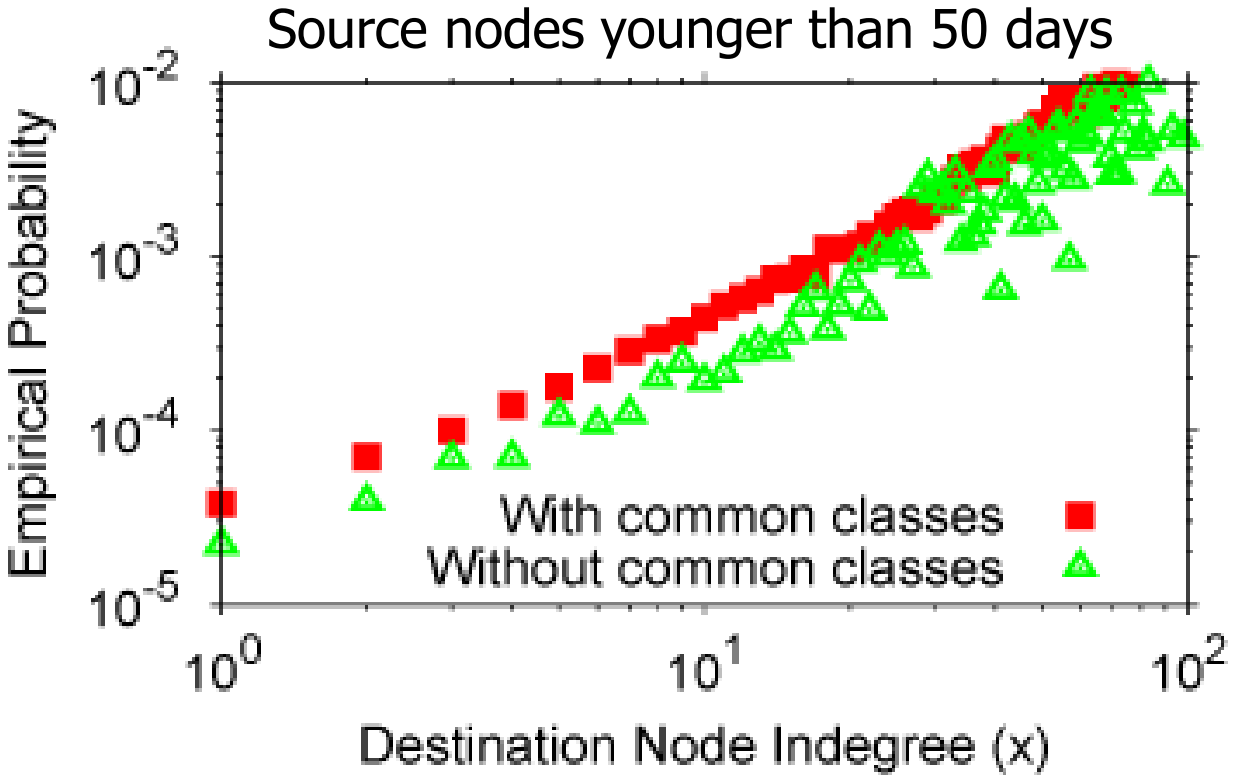


B follows C
B is <u>follower</u> of C
C is <u>friend</u> of B

[Garg et al. IMC '09]

Group Affiliation & Link Formation

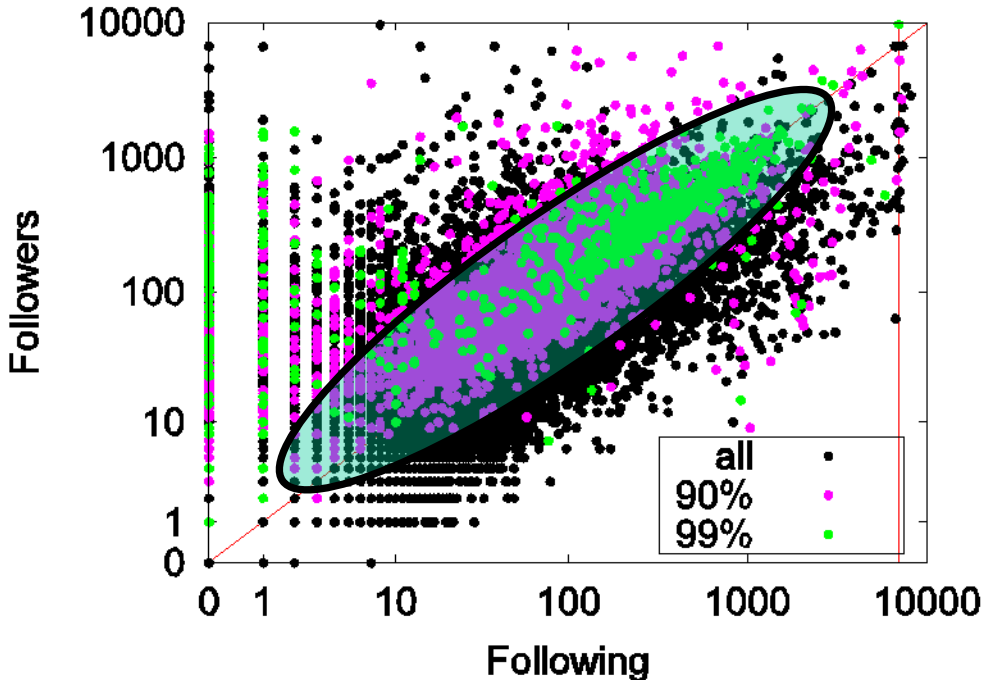
- Does PA explain the observed data? Yes!
- Does subscription to common services (common interest) biases the preference? Yes!



A few chirps about Twitter

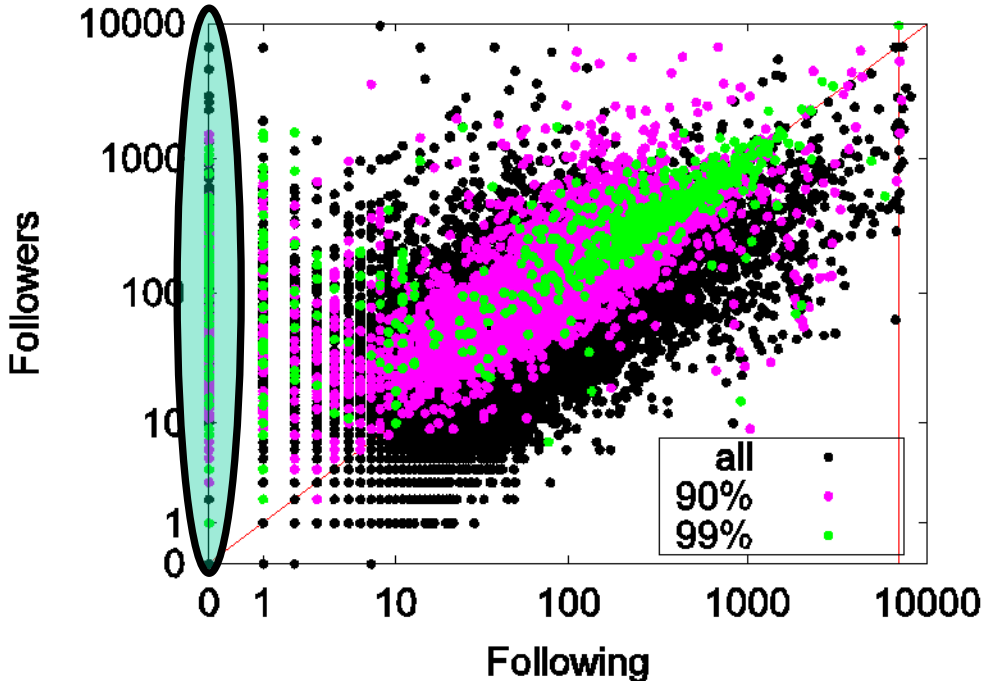
- … by Krishnamurthy, Gill, and Arlitt

Aside: User relationships on Twitter



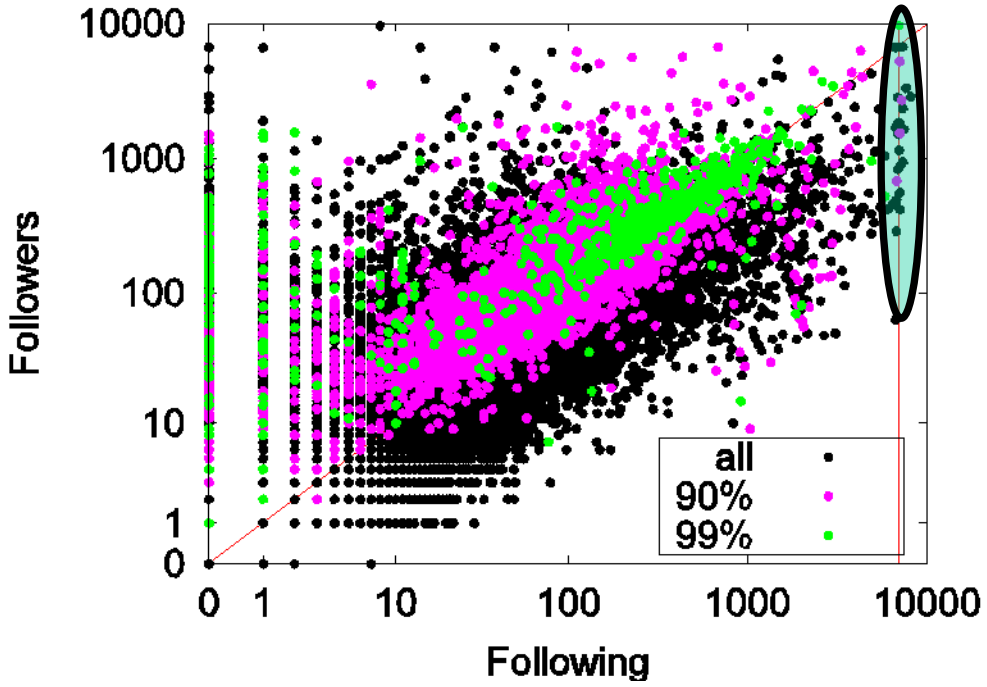
- **Acquaintances**
- Similar number of followers and following
- Along the diagonal
- Green portion is top 1-percentile of tweeters

Aside: User relationships on Twitter



- **Broadcasters**
- News outlets, radio stations
- No reason to follow anyone
- Post playlists, headlines

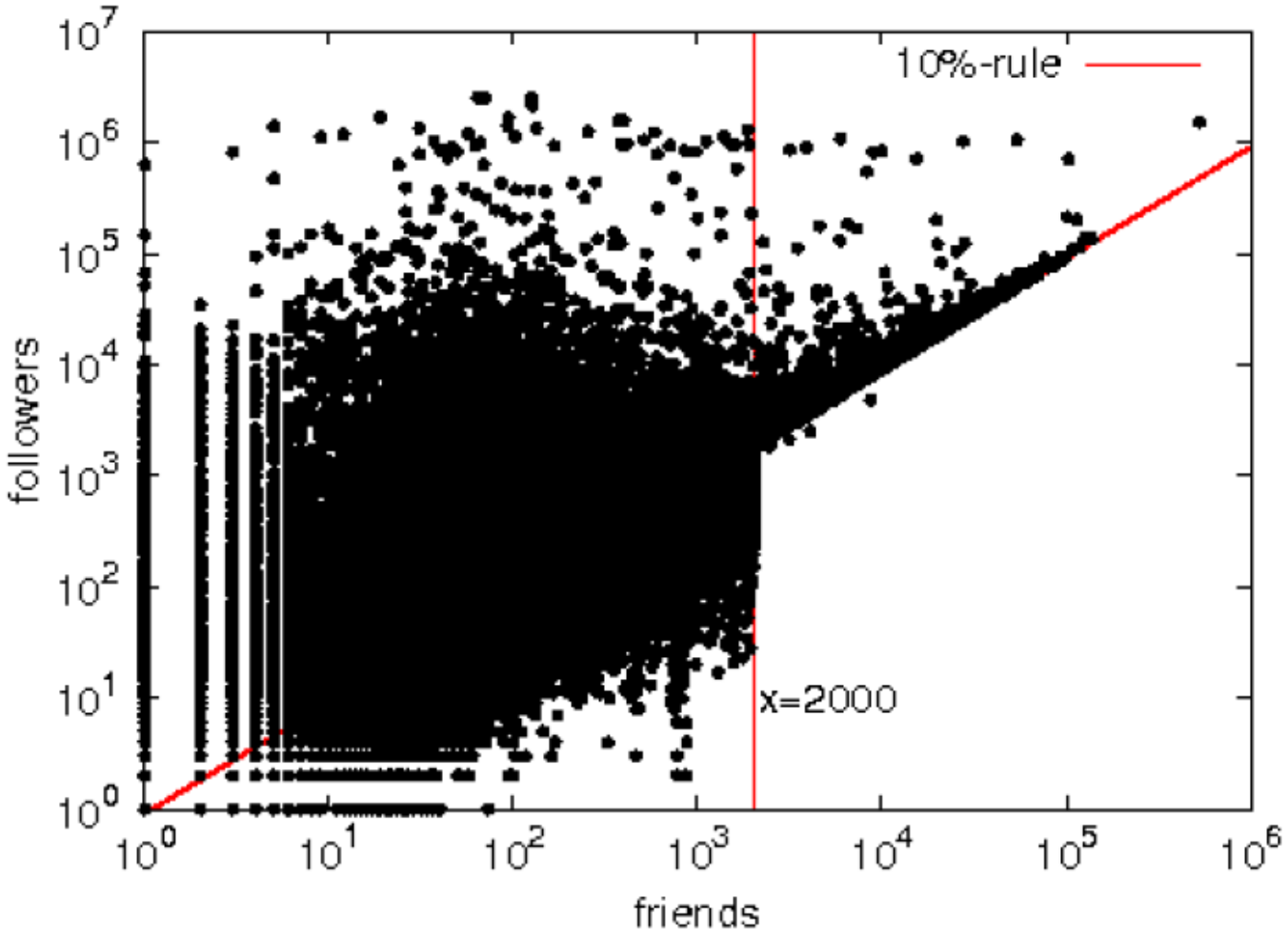
Aside: User relationships on Twitter



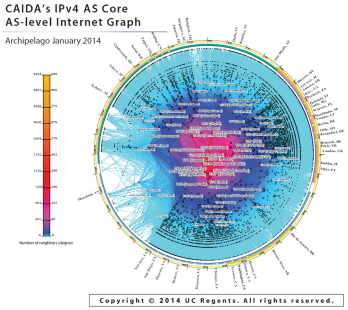
- **Miscreants?**
- Some people follow many users (programmatically)
- Hoping some will follow them back
- Spam, widgets, celebrities (at top)

Aside: User relationships on Twitter

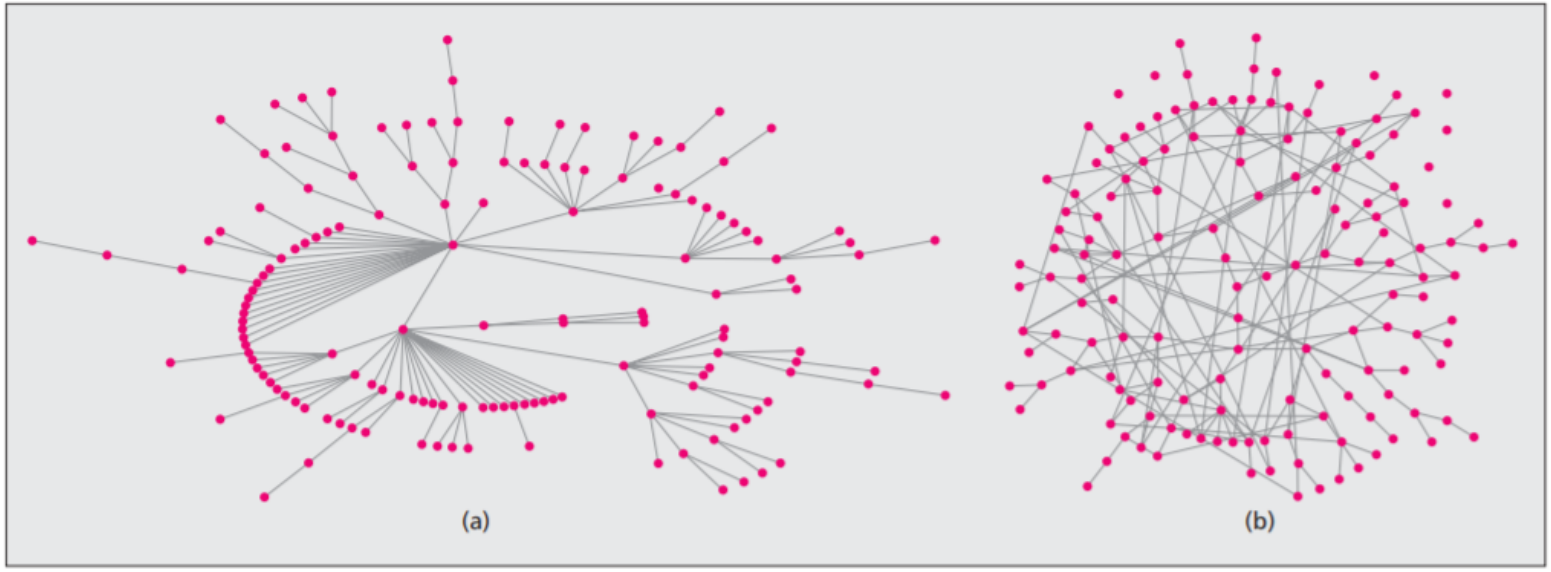
Twitter noticed the miscreants…
… enacted the 10% rule (you can follow 10% more people than follow you)



Are Scale-Free Networks Better?



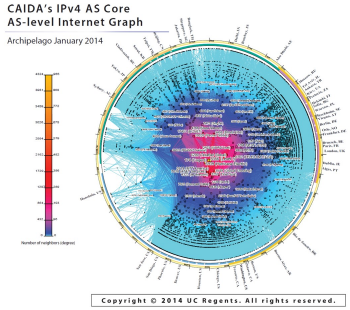
- Scale-free networks have power-law degree distribution (at least asymptotically)
- Average diameter lower in scale-free (SF) than in exponential (E) graphs



Mahanti et al. 2013

Figure 5. *Comparison of power-law and random graphs: Each graph consists of 150 vertices. A vertex is represented by a red dot and the edge is shown using a solid grey line. The graphs were simulated using the NetworkX package in Python, and visualized using* `Graphviz`.

Are Scale-Free Networks Better?



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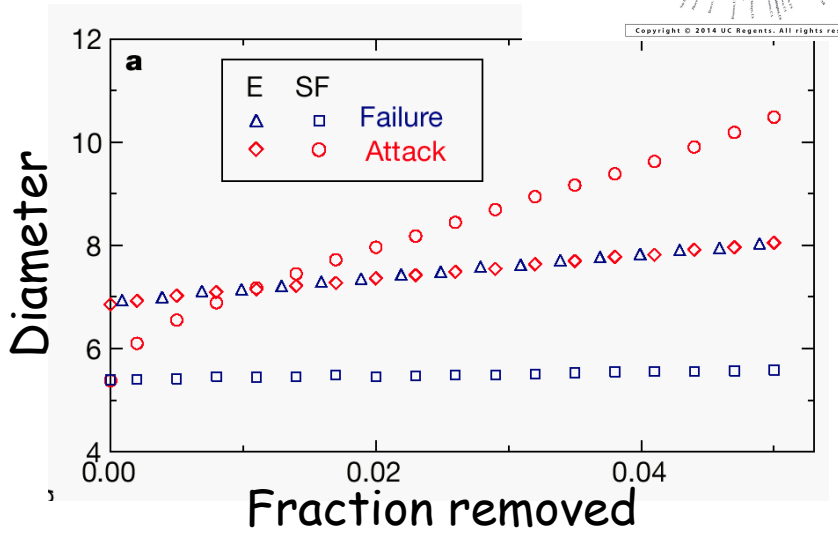

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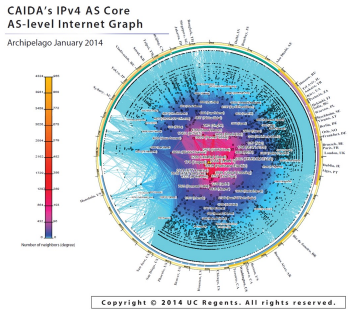

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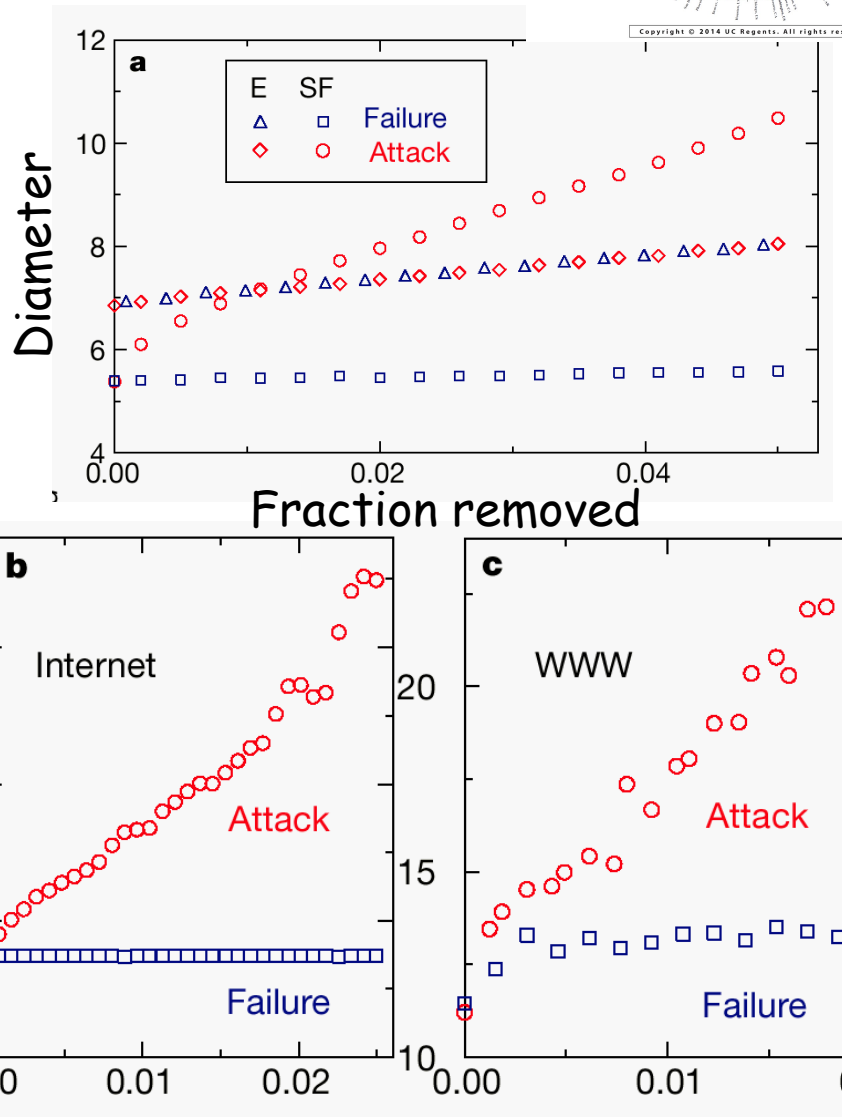
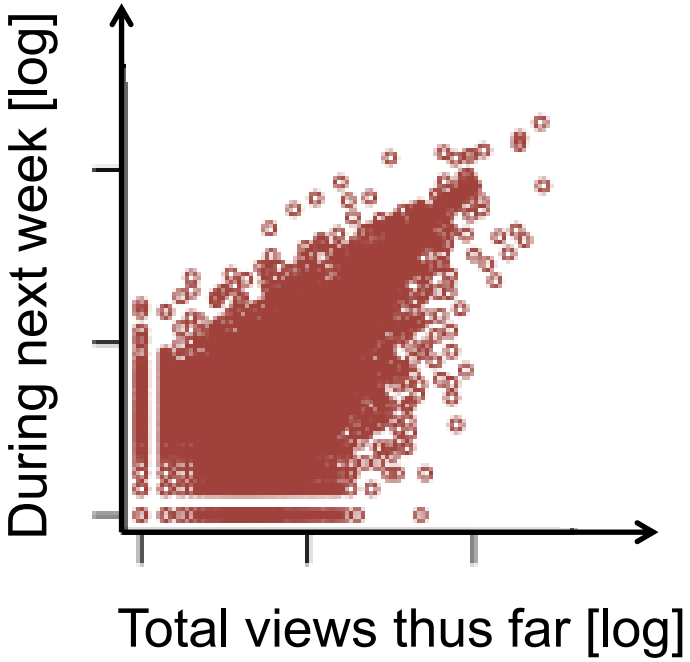
- Same results apply using sampled Internet and WWW graphs (that happen to be scale-free)



… and back to the video example again …

E.g., ACM KDD ’12, IFIP Performance ‘11, ACM TWEB

Rich-gets-richer …
… and churn



E.g., Borghol et al.

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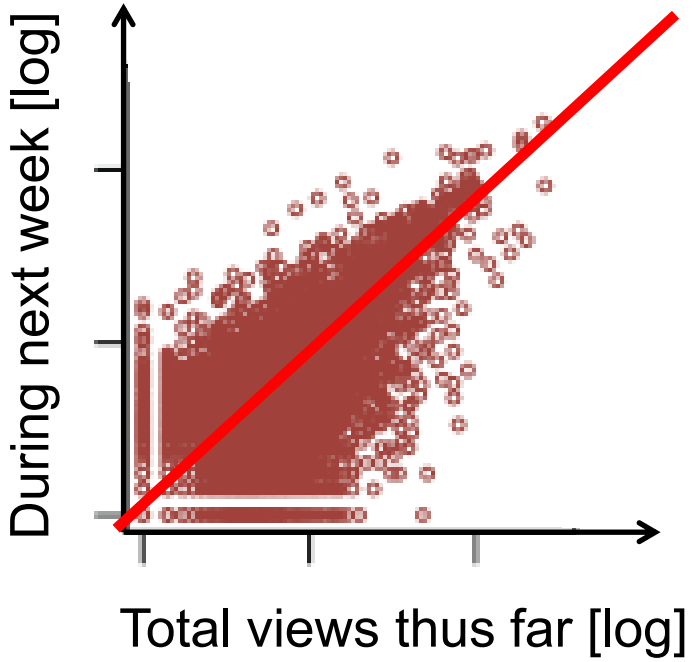

- The more views a video has, the more views it is likely to get in the future

E.g., Borghol et al.

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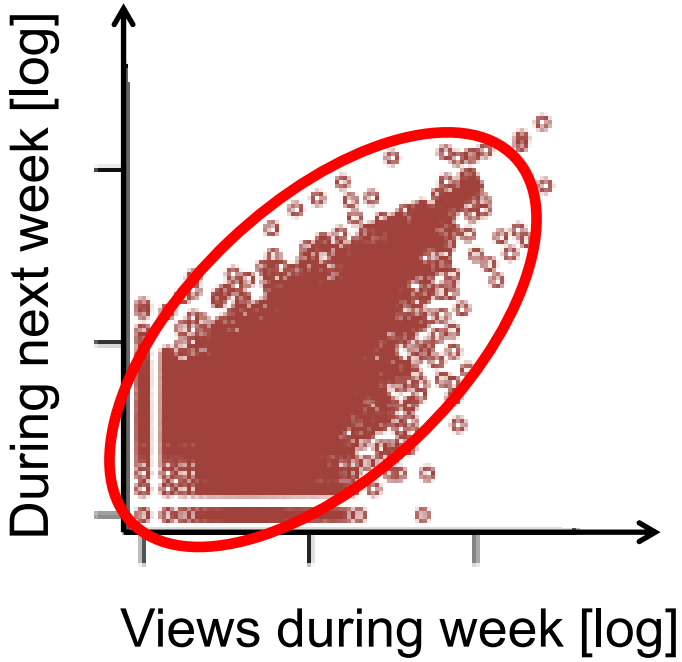

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- The relative popularity of the individual videos are highly non-stationary

E.g., Borghol et al.

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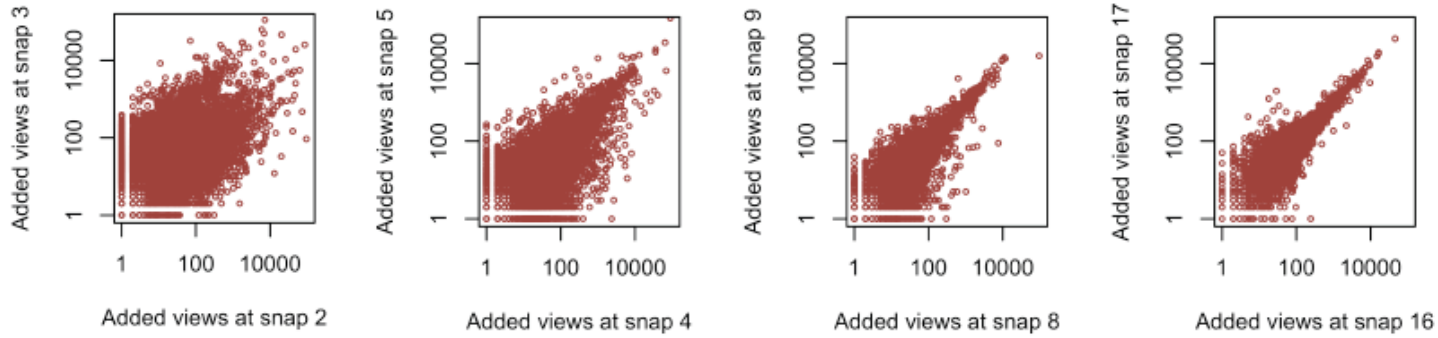

Week 2 Week4 Week 8 Week 16

Young videos Old videos

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E.g., Borghol et al.
IFIP Performance '11

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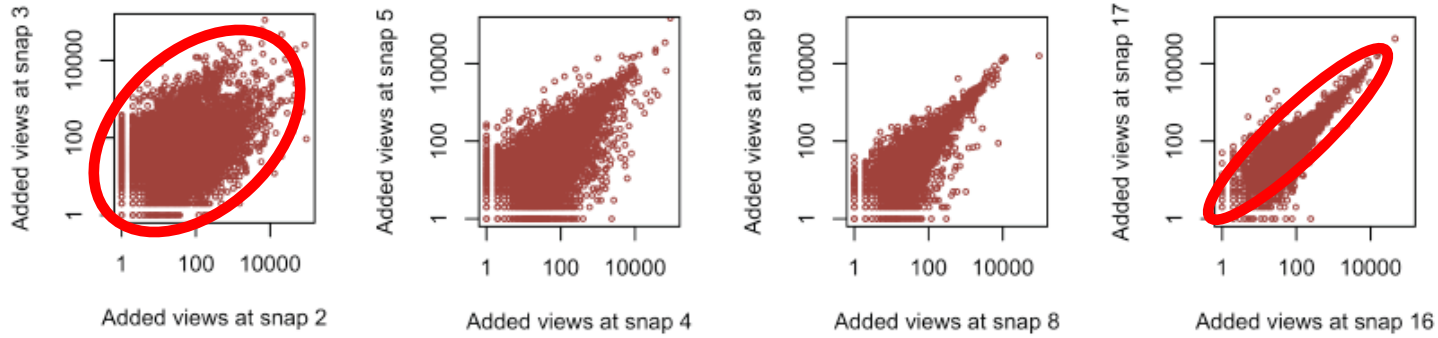

Week 2 Week4 Week 8 Week 16

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E.g., Borghol et al.

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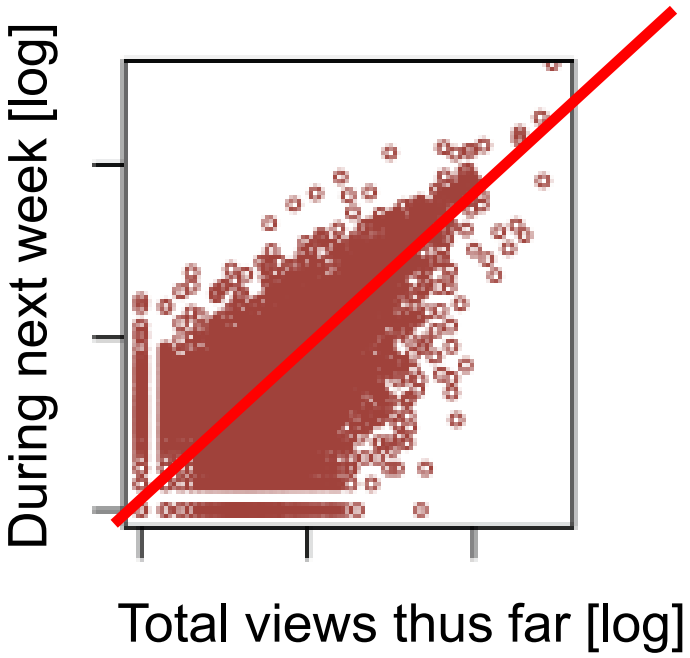

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