Problem Set for Tutorial 2 — TDDD14/TDDD85

1 Basic Concepts

Exercise 1. Determine which of the strings below belong to the language $L(M_1)$ (the DFA is given in Figure 1). Choose one of the accepted strings and prove (see Definition 3 in the lecture manuscript) that it is accepted. Also give $L(M_1)$ in set notation.

1. 001,
2. 001011011011,
3. 00101101001.

Exercise 2. Determine which of the strings below belong to the language $L(M_2)$ (the NFA is given in Figure 2). Also give $L(M_2)$ in set notation.

1. 11110101,
2. 1111111,
3. 101101101111,
4. 10110010011.

Exercise 3. Determine which of the strings below belong to the language $L(M_3)$ (the NFA$_\varepsilon$ is given in Figure 3). Also give $L(M_3)$ in set notation.

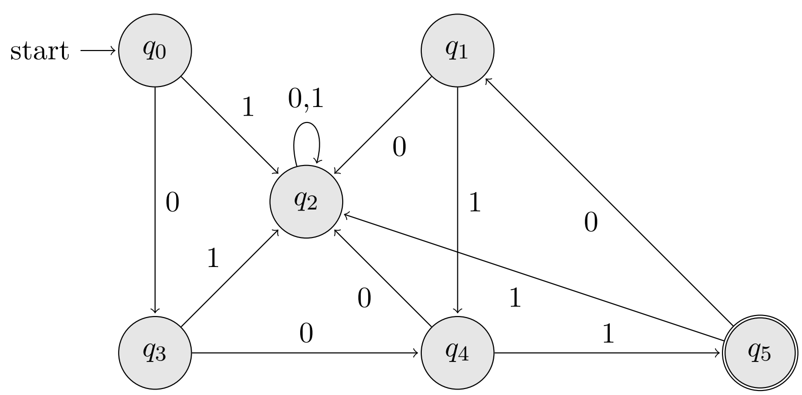


Figure 1: The DFA M_1.

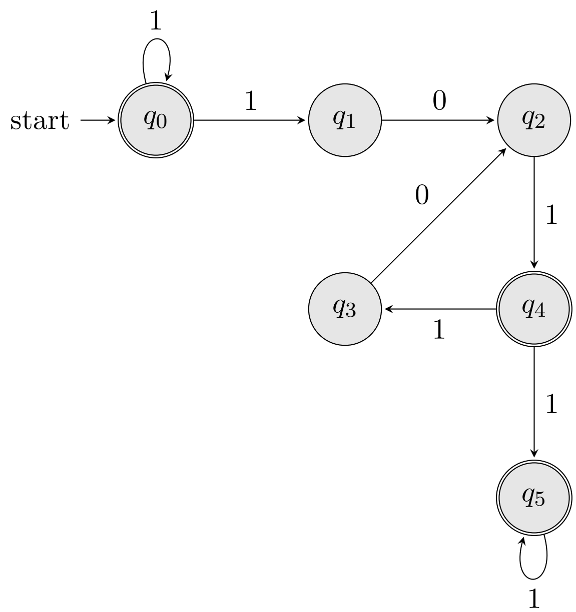


Figure 2: The NFA M_2.

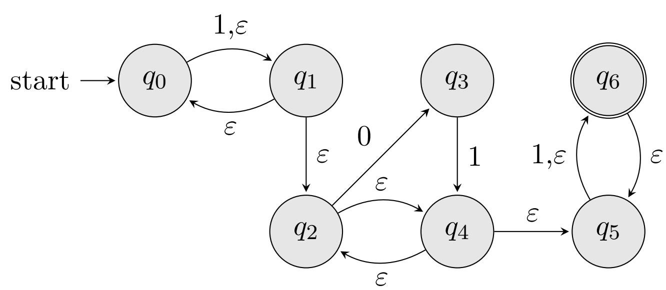


Figure 3: The NFA M_3.

1. 11111,
2. 1101011,
3. 1011100,
4. 0101111.

Exercise 4. For each of the following languages, construct a DFA that accepts the language.

1. $L_1 = \{x \in \{0,1\}^* \mid x \text{ ends in } 00\}$.
2. $L_2 = \{x \in \{0,1\}^* \mid x = (01)^n, n \geq 0\}$.
3. $L_3 = \{x \in \{0,1\}^* \mid \text{ every 0 is immediately followed by 1}\}$.

Solutions

Solution to Exercise 1.

1. Yes, this string is accepted. We begin in q_0, continue to q_3 and q_4, and end up in q_5 after having read the last symbol. Formally, our sequence of states is thus q_0, q_3, q_4, q_5, and we can express the sequence of transitions as: $\delta(q_0, 0) = q_3$, $\delta(q_3, 0) = q_4$, $\delta(q_4, 1) = q_5$, and we accept since q_5 is an accept state.

2. Yes.

3. No (we proceed as in the previous string but instead of reading the final 1 to get us to the accept state q_5 halt in q_4.)

To construct $L(M_1)$ we first realize that q_2 is a "trash" state, i.e., once we end up in q_2 there is no going back. When starting in q_0 the only possible string that we can start with is 001. We are then either done, or can continue by reading 011, 011011, and so on, since any other transition goes to q_2. Thus,

$$L(M_1) = \{001\}\{011\}^*.$$

Solution to Exercise 2.

1. No.

2. Yes (by repeatedly looping in q_0 which is both a start and an accept state).

3. Yes.

4. No.

We can either read an arbitrary number of 1s and stay in the start state, which is also an accept state. Otherwise we leave this state by reading 1, followed by 0, and 1. We can then loop in q_4 by reading 101, or go to q_5 where we can read an arbitrary number of 1s. Put together the language can be described as $L(M_3) = \{1\}^* \cup \{1\}^+\{01\}\{101\}^*\{1\}^*$.

Solution to Exercise 3.

1. Yes.

2. Yes.

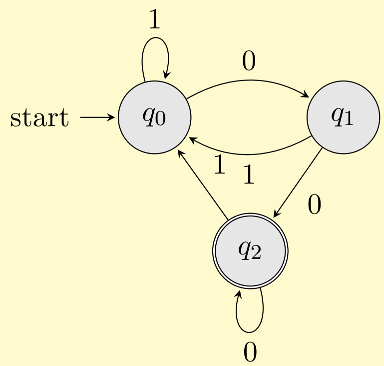
3. No.

4. Yes.

This is a pretty nasty machine but which still recognizes a rather simple language. The behaviour is roughly that we (1) read an arbitrary number of 1s, optionally read 01 how many times that we want, and finish in q_6 by again reading an arbitrary number of 1s. This gives us $L(M_3) = \{1\}^*\{01\}^*\{1\}^*$.

Solution to Exercise 4.

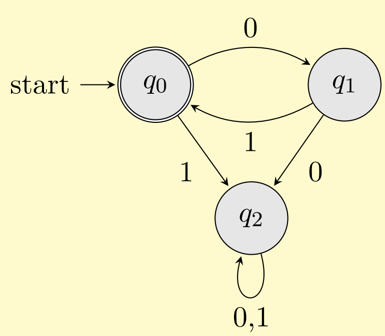
1. An example of a DFA M_{15} such that $L(M_{15}) = L_1$ is given here.



Specification of the states:

- q_0: The last symbol read, if any, is 1.
- q_1: The last symbol read is 0; the last but one, if any, is 1.
- q_2: The last two symbols read are 00.

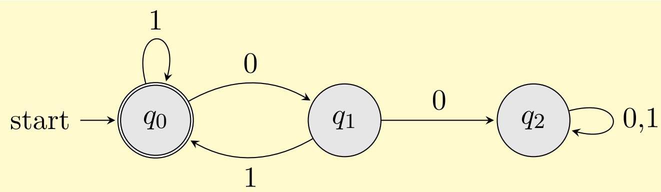
2. An example of a DFA M_{16} such that $L(M_{16}) = L_2$ is given here.



Specification of the states:

- q_0: Any number (incl. 0) of 01s read.
- q_1: Any number (incl. 0) of 01s followed by 0 read.
- q_2: Something else read.

3. An example of a DFA M_{17} such that $L(M_{17}) = L_4$ is given here.



Specification of the states:

- q_0: The last symbol read, if any, is 1; any previous 0 is immediately followed by 1.
- q_1: The last symbol read is 0, any previous 0 is immediately followed by 1.
- q_2: 00 has been read.

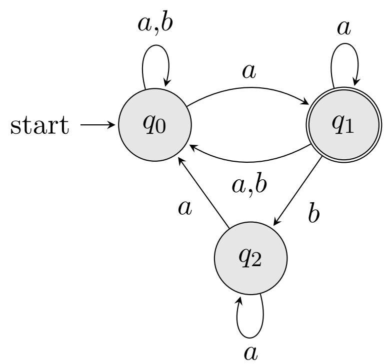


Figure 4: The NFA M_4.

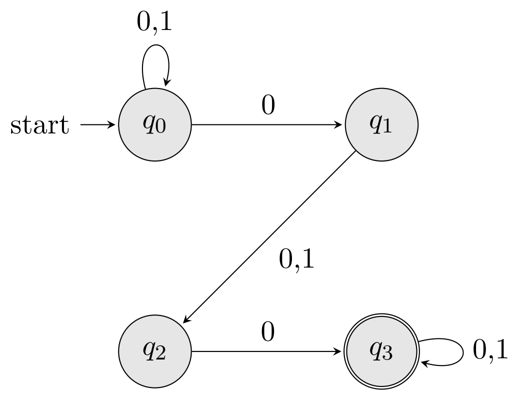


Figure 5: The NFA M_5.

2 The Subset Construction

Exercise 5. Two automata M and M' are equivalent if they accept the same language, i.e., $L(M) = L(M')$.

1. Given the NFA in Figure 4, construct an equivalent DFA.
2. Given the NFA in Figure 5, construct an equivalent DFA.
3. Given the NFA in Figure 6, construct an equivalent DFA.

Exercise 6.

1. Given the NFA$_\varepsilon$ in Figure 7, construct an equivalent DFA.
2. Given the NFA$_\varepsilon$ in Figure 8, construct an equivalent DFA.

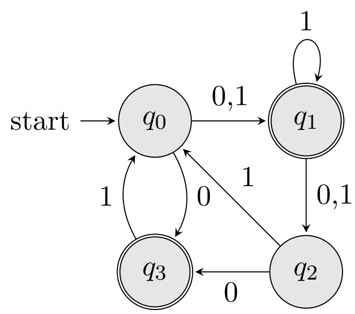


Figure 6: The NFA M_6.

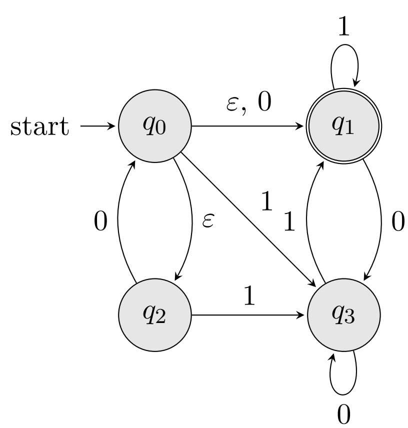


Figure 7: The NFA$_\varepsilon$ M_7

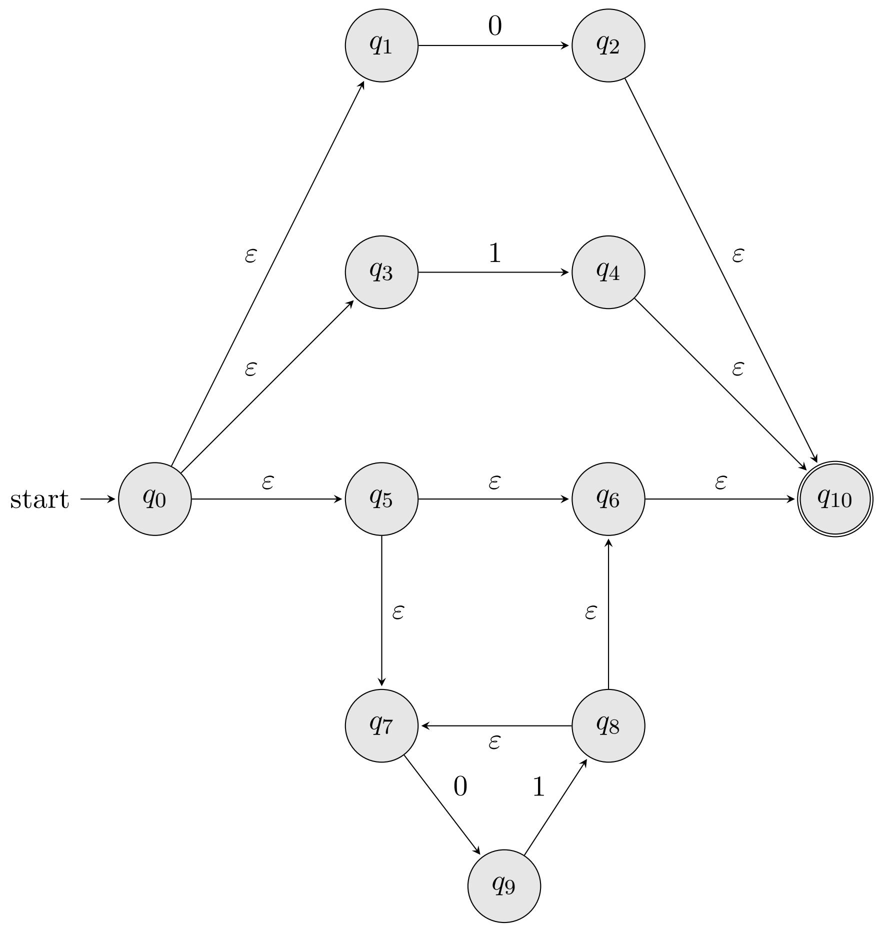


Figure 8: The NFA$_\varepsilon$ M_8

Solutions

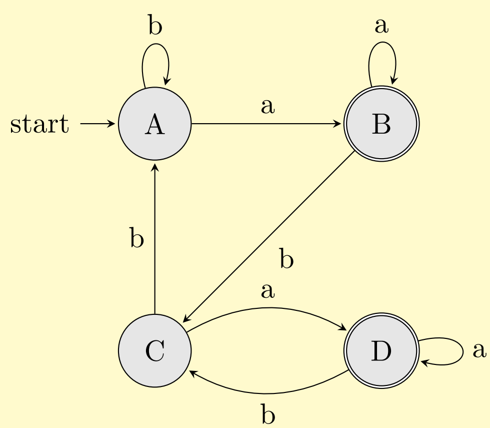
Solution to Exercise 5. We often use [] instead of {} to denote a set of states which is a state of a DFA.

1. δ_{18} is given here.

State	a	b
$[q_0]$	$[q_0, q_1]$	$[q_0]$
$[q_0, q_1]$	$[q_0, q_1]$	$[q_0, q_2]$
$[q_0, q_2]$	$[q_0, q_1, q_2]$	$[q_0]$
$[q_0, q_1, q_2]$	$[q_0, q_1, q_2]$	$[q_0, q_2]$

The new set of final states is $F' = \{x \in 2^Q \mid x \cap \{q_1\} \neq \emptyset\} = \{\{q_0, q_1\}, \{q_0, q_1, q_2\}\}$.

Let $[q_0] = A$, $[q_0, q_1] = B$, $[q_0, q_2] = C$, $[q_0, q_1, q_2] = D$. The transition diagram for the DFA is then as follows.



2. δ_{19} is given here.

State	0	1
$[q_0]$	$[q_0, q_1]$	$[q_0]$
$[q_0, q_1]$	$[q_0, q_1, q_2]$	$[q_0, q_2]$
$[q_0, q_1, q_2]$	$[q_0, q_1, q_2, q_3]$	$[q_0, q_2]$
$[q_0, q_2]$	$[q_0, q_1, q_3]$	$[q_0]$
$[q_0, q_1, q_2, q_3]$	$[q_0, q_1, q_2, q_3]$	$[q_0, q_2, q_3]$
$[q_0, q_2, q_3]$	$[q_0, q_1, q_3]$	$[q_0, q_3]$
$[q_0, q_1, q_3]$	$[q_0, q_1, q_2, q_3]$	$[q_0, q_2, q_3]$
$[q_0, q_3]$	$[q_0, q_1, q_3]$	$[q_0, q_3]$

The new set of final states is

$$F' = \{x \in 2^Q \mid x \cap \{q_3\} \neq \emptyset\} = \{[q_0, q_3], [q_0, q_1, q_3], [q_0, q_2, q_3], [q_0, q_1, q_2, q_3]\}.$$

We leave the construction of the transition diagram for the DFA to the reader.

3. δ_{20} is given here.

State	0	1
$[q_0]$	$[q_1, q_3]$	$[q_1]$
$[q_1, q_3]$	$[q_2]$	$[q_0, q_1, q_2]$
$[q_1]$	$[q_2]$	$[q_1, q_2]$
$[q_2]$	$[q_3]$	$[q_0]$
$[q_0, q_1, q_2]$	$[q_1, q_2, q_3]$	$[q_0, q_1, q_2]$
$[q_1, q_2]$	$[q_2, q_3]$	$[q_0, q_1, q_2]$
$[q_3]$	$\emptyset$	$[q_0]$
$[q_1, q_2, q_3]$	$[q_2, q_3]$	$[q_0, q_1, q_2]$
$[q_2, q_3]$	$[q_3]$	$[q_0]$
$\emptyset$	$\emptyset$	$\emptyset$

The new set of final states is $F' = \{x \in 2^Q \mid x \cap \{q_1, q_3\} \neq \emptyset\} =$ $\{[q_1, q_3], [q_1], [q_0, q_1, q_2], [q_1, q_2], [q_3], [q_1, q_2, q_3], [q_2, q_3]\}$ We leave the construction of the transition diagram for the DFA to the reader.

Solution to Exercise 6.

1. We apply the subset construction to an NFA$_\varepsilon$. Thus, whenever a state P of the contructed DFA contains a state q (of the NFA$_\varepsilon$), all the states reachable from q by ϵ-transitions (in the NFA$_\varepsilon$) are also in P.

   The following table gives the transition function δ_{21} of a DFA corresponding to the NFA$_\varepsilon$ from Figure 7. The initial state is $\{q_0, q_1, q_2\}$. All the states are final except $\{q_3\}$.

State	0	1
$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2, q_3\}$	$\{q_1, q_3\}$
$\{q_1, q_3\}$	$\{q_3\}$	$\{q_1\}$
$\{q_1\}$	$\{q_3\}$	$\{q_1\}$
$\{q_3\}$	$\{q_3\}$	$\{q_1\}$
$\{q_0, q_1, q_2, q_3\}$	$\{q_0, q_1, q_2, q_3\}$	$\{q_1, q_3\}$

2. The subset construction results in an DFA with reachable states $\{q_0, q_1, q_3, q_5, q_6, q_7, q_{10}\}$, $\{q_2, q_9, q_{10}\}$, $\{q_4, q_{10}\}$, $\emptyset$, $\{q_6, q_7, q_8, q_{10}\}$, $\{q_9\}$. The initial state is $\{q_0, q_1, q_3, q_5, q_6, q_7, q_{10}\}$. The final states are those containing q_{10}.

3 Advanced and Exam Like Exercises

Exercise 7. For each k give an example of an NFA with $\leq k+1$ states where the subset construction yields a DFA with 2^k states.

Exercise 8. The NFA with ϵ transitions N is defined via the transition function

State	ε	a	b
$\rightarrow$ 1	$\{2\}$	$\{3\}$	$\{3\}$
2	$\emptyset$	$\{1,4\}$	$\{3\}$
3 F	$\{4\}$	$\emptyset$	$\emptyset$
4	$\emptyset$	$\emptyset$	$\{2\}$

Recall that $\rightarrow$ indicates the start state, and that F indicates an accept/final state.

1. Draw the transition diagram for N.
2. Using a standard method, construct an equivalent DFA M.

Exercise 9. Give DFAs or NFA$_\varepsilon$s for each of the following subsets of $\{0,1\}^*$. Clearly state any assumptions you make.

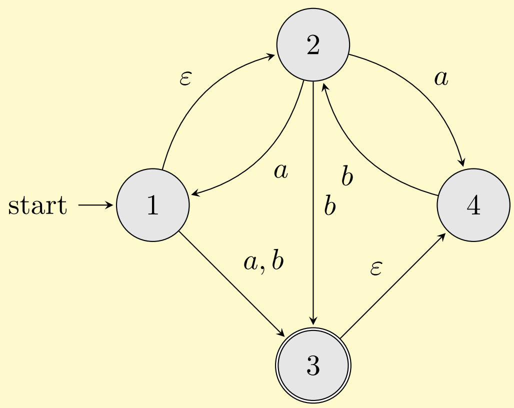
1. $\{x \mid x$ contains an even number of 0's$\}$,
2. $\{x \mid x$ contains an odd number of 1's$\}$,
3. $\{x \mid x$ contains an even number of 0's or an odd number of 1's$\}$,
4. $\{x \mid x$ contains two consecutive 0's but not three consecutive 0's$\}$.

Solutions

Solution to Exercise 7. There are many such examples. Here is one of them which we illustrate for the case $k = 3$. Let $\Sigma = \{a, b, c\}$ and consider the language $\{a, b\}^* \cup \{a, c\}^* \cup \{b, c\}^*$, i.e., the set of all strings formed by at most 2 of the symbols in Σ. This language can be recognized by an NFA with 4 states whose starting state guesses the missing letter via ε-transitions. However, the smallest DFA which accepts this language has $2^3 = 8$ states. This is not so easy to prove via a direct argument but follows from the Myhill-Nerode theorem which comes later in the course.

For a general answer we repeat the above construction but with an alphabet with k symbols.

Solution to Exercise 8.



1. See the NFA$_\varepsilon$ here.
2. By using the subset construction one should obtain a DFA with 7 states, of which 3 are accept states, and one a "garbage" state.

Solution to Exercise 9. A potential solution was discussed at the lecture. Hint: for the first two tasks, remember that we only need to count modulo 2 in order to check whether we have an even or odd number of a symbol. For the third task, you already know how to construct a DFA or NFA for recognizing an even number of 0s or an odd number of 1s. Try to construct an NFA which combines these two machines into a single machine.