4 Regular Languages

4.1 Let $\Sigma = \{0, 1\}$ and $\Gamma = \{a, b\}$ be two alphabets. Suppose $h : \Sigma \to \Gamma^*$ is a homomorphism such that:

$$h(0) = aaab$$
$$h(1) = bbba$$

a) What is $h(01)$?

b) What is $h(101)$?

c) Suppose $L \subseteq \Sigma^*$ is the language 001^*. Give a regular expression for $h(L)$.

4.2 Let $\Sigma = \{0, 1\}$ and $\Gamma = \{a, b\}$ be two alphabets. Suppose $h : \Sigma \to \Gamma^*$ is a homomorphism such that:

$$h(0) = aa$$
$$h(1) = aba$$

Suppose $L \subseteq \Gamma^*$ is the language $(ab + ba)^*a$. Give a *minimal* DFA which accepts $h^{-1}(L) = \{x \in \Sigma^* \mid h(x) \in L\}$.

4.3 Suppose $L \subseteq \{a, b\}^*$ is the language $(ab)^* + a(ba + a)^*$. Give a *minimal* DFA which accepts the language $\overline{L} = \{x \in \{a, b\}^* \mid x \notin L\}$.

4.4 Suppose $L_1 \subseteq \{0, 1\}^*$ and $L_2 \subseteq \{0, 1\}^*$ are the languages $(0 + 11)^*$ and 11^*0 respectively. Give a *minimal* DFA which accepts the language $L_1 \cap L_2$.

4.5 Show, by using the pumping lemma, that the following languages are *not* regular.

a) $L_1 = \{0^n 1^n \mid n \geq 0\}$

b) $L_2 = \{x \in \{0, 1\}^* \mid x = x^R\}$

4.2 Since $L \subseteq \Gamma^*$ is a regular language, there exists a DFA $M_{26} = (Q, \Gamma, \delta_{26}, q_0, F)$ which accepts L (see figure 26). (Notice that there exist simpler DFA's for L.)

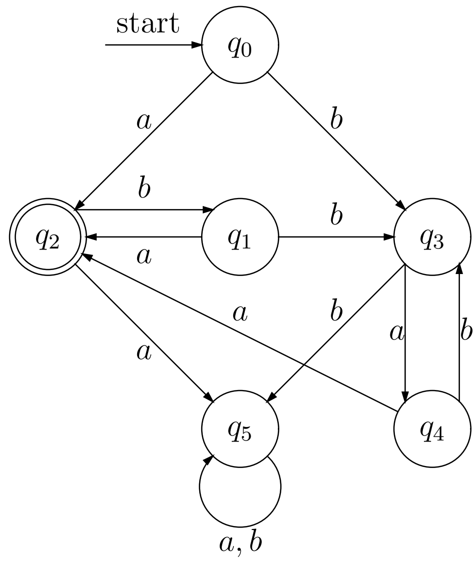

Figure 26: M_{26}

A new DFA M', accepting $h^{-1}(L)$, may now be constructed from M_{26} as follows: Let $M' = (Q, \Sigma, \delta_{27}, q_0, F) = (\{q_0, q_1, q_2, q_3, q_4, q_5\}, \{0, 1\}, \delta_{27}, q_0, \{q_2\})$ where δ_{27} is defined by $\delta_{27}(q, x) = \delta_{26}(q, h(x))$ for $q \in Q$ and $x \in \{0, 1\}$, e.g. $\delta_{27}(q_0, 0) = \delta_{26}(q_0, h(0)) = \delta_{26}(q_0, aa) = q_5$ (see table 6).

State	*Input*	
	0	1
q_0	q_5	q_2
q_1	q_5	q_2
q_2	q_5	q_5
q_3	q_2	q_4
q_4	q_5	q_2
q_5	q_5	q_5

Table 6: δ_{27}

The only states which can be reached from q_0 are q_0, q_2, and q_5. Thus the states q_1, q_3, and q_4 may be removed from M'. The result is the DFA M_{27}, which accepts $h^{-1}(L)$ and which can be shown to be minimal (see figure 27).

4.3 L is the language $(ab)^* + a(ba + a)^*$. Let $M = (Q, \{a, b\}, \delta, q_0, F)$ be a DFA such that $L(M) = L$. Then the DFA $M' = (Q, \{a, b\}, \delta, q_0, F')$, where $F' = Q - F$, accepts $\overline{L}$. A minimal DFA M_{28} is given in figure 28.

Note that if M is minimal then M' is minimal too. This is because the equivalence relation $\equiv_{...}$ is the same for both languages ($\equiv_L = \equiv_{\overline{L}}$).

4.4 $L_1 = (0 + 11)^*$ and $L_2 = 11^*0$. DFAs $M_{29} = (Q_1, \{0, 1\}, \delta_1, q_{1,0}, F_1)$ accepting L_1 and $M_{30} = (Q_2, \{0, 1\}, \delta_2, q_{2,0}, F_2)$ accepting L_2 are given in figure 29 and figure 30 respectively.

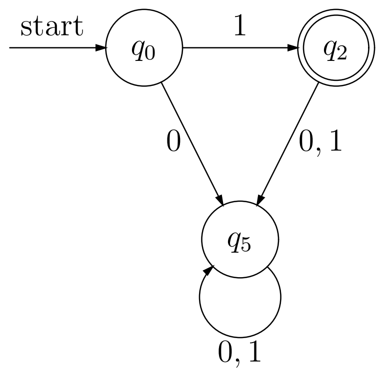

Figure 27: M_{27}

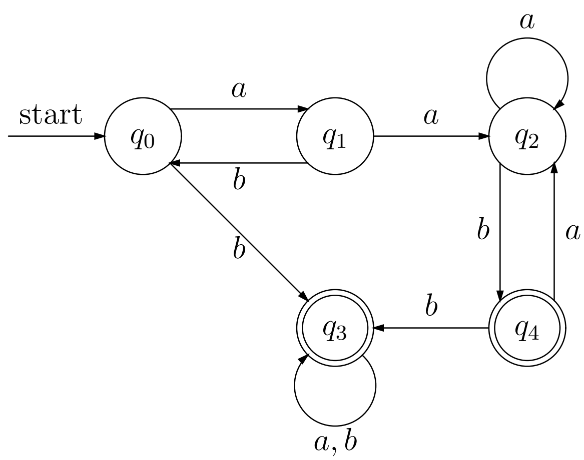

Figure 28: M_{28}

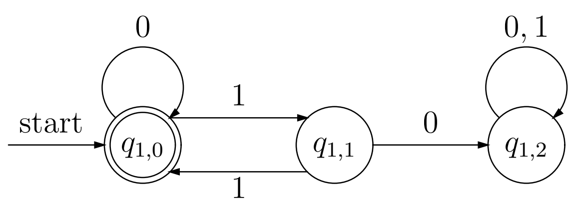

Figure 29: M_{29}

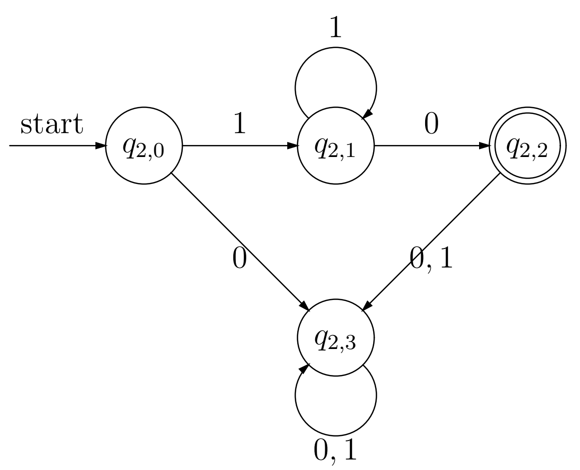

Figure 30: M_{30}

A DFA M accepting $L_1 \cap L_2$ may now be constructed from M_{29} and M_{30} as follows: Let $M = (Q_1 \times Q_2, \{0,1\}, \delta_7, \langle q_{1,0}, q_{2,0}\rangle, F_1 \times F_2)$. The states of M are thus pairs where the first component is a state of M_1 and the second component is a state in M_2. The transition function δ_7 is then defined by $\delta_7(\langle q, r\rangle, x) = \langle \delta_1(q,x), \delta_2(r,x)\rangle$ for $q \in Q_1$, $r \in Q_2$ and $x \in \{0,1\}$ (see table 7).

State	*Input*	
	0	1
$\langle q_{1,0}, q_{2,0}\rangle$	$\langle q_{1,0}, q_{2,3}\rangle$	$\langle q_{1,1}, q_{2,1}\rangle$
$\langle q_{1,0}, q_{2,3}\rangle$	$\langle q_{1,0}, q_{2,3}\rangle$	$\langle q_{1,1}, q_{2,3}\rangle$
$\langle q_{1,1}, q_{2,1}\rangle$	$\langle q_{1,2}, q_{2,2}\rangle$	$\langle q_{1,0}, q_{2,1}\rangle$
$\langle q_{1,1}, q_{2,3}\rangle$	$\langle q_{1,2}, q_{2,3}\rangle$	$\langle q_{1,0}, q_{2,3}\rangle$
$\langle q_{1,2}, q_{2,2}\rangle$	$\langle q_{1,2}, q_{2,3}\rangle$	$\langle q_{1,2}, q_{2,3}\rangle$
$\langle q_{1,0}, q_{2,1}\rangle$	$\langle q_{1,0}, q_{2,2}\rangle$	$\langle q_{1,1}, q_{2,1}\rangle$
$\langle q_{1,2}, q_{2,3}\rangle$	$\langle q_{1,2}, q_{2,3}\rangle$	$\langle q_{1,2}, q_{2,3}\rangle$
$\langle q_{1,0}, q_{2,2}\rangle$	$\langle q_{1,0}, q_{2,3}\rangle$	$\langle q_{1,1}, q_{2,3}\rangle$

Table 7: δ_7

Minimization of M (and renaming of states) gives M_{31} of figure 31.

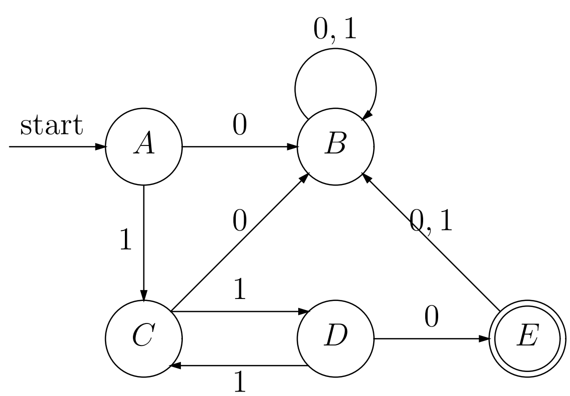

Figure 31: M_{31}

4.5 The pumping lemma: If L is a regular language, then there exists a constant n such that $z \in L$ and $|z| \geq n$ implies that there exist strings u, v and w satisfying the following conditions:

1. $z = uvw$
2. $|v| \geq 1$ and $|uv| \leq n$
3. $uv^iw \in L$ for all $i \geq 0$

In order to show that a language is *not* regular, we first assume the opposite to be true, and then use the pumping lemma to show that our assumption leads to a contradiction.

a) Suppose L_1 is a regular language and let n be the constant that then exists according to the lemma. Consider the string $z = 0^n 1^n$. We have $z \in L_1$ and $|z| = 2n \geq n$, which implies that there should be strings u, v and w satisfying the above conditions. If we can show that there in fact are no such strings, i.e. that regardless of how u, v and w are chosen, $uv^iw \notin L_1$ for some i, then we have obtained our contradictions.

   $z = uvw$ and $|uv| \leq n$ implies that the strings u and v can only consist of 0, i.e. $u = 0^k$ and $v = 0^l$ where $k + l \leq n$. This means that we must have $w = 0^{n-k-l}1^n$.

   Now consider $uv^2w = 0^{n+l}1^n$. If $|v| \geq 1$, then this implies $l \geq 1$, i.e. $uv^2w \notin L_2$. Thus we have a contradiction.

b) Suppose L_2 is a regular language and let n be the constant in the pumping lemma. Then consider $z = 0^n 1 0^n$; $z \in L_2$ and $|z| = 2n + 1 \geq n$. $z = uvw$ and $|uv| \leq n$ implies that $u = 0^k$, $v = 0^l$ $(k + l \leq n)$ and $w = 0^{n-k-l}10^n$.

   Now consider $uv^0w = uw = 0^{n-l}10^n$. Since $|v| \geq 1 \Rightarrow l \geq 1$, we have $n - l < n$, i.e. $uv^0w = uw \notin L_2$. Contradiction!