Tentamen
TDDB56 DALGOPT-D Algoritmer och Optimering
14 december 2002, kl 8-13

Lokal: GARN, Garnisonen, Linköping
Examinator: Hans Olsén
Jourhavande lärare:
Hans Olsén (013-282365, 070-2958795)
Kaj Holmberg (013-282867)
Visning: date/time to be announced on the course homepage

Hjälpmedel / Admitted material:
Miniräknare / pocket calculator
Engelsk-Svensk ordbok / Swedish-English dictionary

Kursböckerna DALG:
- Lewis, Denenberg: Data Structures and Their Algorithms.
- Goodrich, Tamassia: Data Structures and Algorithms in Java.

Årets kurslitteratur OPT:
- Papadimitriou, Steiglitz: Combinatorial Optimization,

Kurslitteratur för TAOP13 (2001):
- Linjär och ickelinjär optimering, Lundgren, Rönnqvist och Värbrand.
- Flöden i nätverk och kombinatorisk optimering, Holmberg.
- Kombinatorisk optimering II, Holmberg.
- Utdrag ur Kombinatorisk Optimering, problem och algoritmer, Migdalas and Göthe-Lundgren.

Kurslitteratur för TAOP13 (2000):
- Kombinatorisk Optimering, problem och algoritmer, Migdalas och Göthe-Lundgren.
General instructions

- This exam has 9 assignments and 6 pages in total. Read all assignments carefully and completely before you begin.
- Use a separate sheet for each assignment. Mark each sheet on top with your name, personnummer, and the course code (TDDB56).
- You may answer in either English or Swedish.
- Write clearly. Unreadable text will be ignored.
- Be precise in your statements. Unprecise formulations may lead to a reduction of points.
- Motivate clearly all statements and reasoning.
- Explain calculations and solution procedures.
- The assignments are not ordered according to difficulty.
- If you use some result from the admitted literature that has not been part of the lectures, give a reference (book title, page) or give an explanation.
- One copy of each of the admitted course books Lewis/Denenberg and Goodrich/Tamassia is available for inspection at the central tentavakt desk. Please don’t remove them from there.

1. (3p) Which of the following statements are true and which are false? Justify your answers. Answer without justification gives no credits.

(a) $n^2 \in O(2^n)$
(b) $n \log n \in \Theta(n)$
(c) $\sqrt{n} \in \Omega(\log n)$
(d) $O(n \log n) \subseteq O(n^2)$
(e) $\Omega(n) \subseteq \Theta(n^2)$
(f) $\Theta(n) \subseteq \Omega(\log n)$
2. (7p) Consider the following LP-problem

\[
\begin{align*}
\text{max } & \quad z = 6x_1 + 3x_2 + 5x_3 \\
\text{subject to } & \quad 3x_1 + x_2 - 2x_3 \leq 3 \\
& \quad 2x_1 + 2x_2 + x_3 \leq 6 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

(a) Formulate the LP-dual to the problem above. Solve the dual graphically. Find primal optimal solution with the help of the complementarity conditions.

(b) Which primal variables are in the optimal basis? Give the basis matrix \(B\).

(c) Give the reduced costs for \(x_1, x_2\) and \(x_3\) at the optimal point.

(d) Start from the origin and do one iteration with the simplex method (in the primal). Which point is obtained? Is it optimal?

3. (3p) Let the set \(S = \{4, 5, 9, 14, 23, 42, 49, 63\}\) be given.

(a) Construct a heap storing the elements of \(S\) (such that the minimum element appears at the root): Draw the tree and the array representation.

(b) Show the corresponding picture after performing a removeMin operation.

(c) Show the corresponding picture after performing an insertItem operation with key \(k=3\) in the original heap.

4. (6p)

(a) How shall a shortest path method (Ford’s or Dijkstra’s) be modified if the graph is undirected? Do the modifications in the algorithm, not in the graph. (We should assume that \(c_{ij} \geq 0 \forall i, j\). Why?)

(b) Find the shortest path from node 1 to node 6 in the undirected graph below.

(c) The arc costs satisfy the triangle inequality if \(c_{ij} \leq c_{ik} + c_{kj} \forall k, i, j\). This means that the direct arc between node \(i\) and node \(j\) (which must exist) never is more expensive than going via another node \(k\).

Describe what happens if you try to solve a shortest path problem with Ford’s method in an undirected graph where the arc costs satisfy the triangle inequality. What do you know about the optimal solution?

(d) Is it possible to correctly solve an undirected shortest path problem by finding the minimal spanning tree and then removing all arcs that does not lie on the path between the starting and ending nodes? Motivate the answer.
5. (7p) Recall that multiplication of two matrices, \( X \) and \( Y \), of dimensions \( p \times q \) and \( q \times r \) respectively (\( \text{no. of rows} \times \text{no. of columns} \)), results in a matrix \( Z = XY \) of dimensions \( p \times r \). You are given the program \texttt{MultiplyMatrices}(X,Y) which computes the product of \( X \) and \( Y \) requiring \( pqr \) scalar multiplications (“skalärmultiplikationer”). That is, the time complexity is \( O(pqr) \). Matrix multiplication is not commutative, but it is associative, which means that \( ABC = (AB)C = A(BC) \). The product \( ABC \) can thus be evaluated either as \texttt{MultiplyMatrices}(\texttt{MultiplyMatrices}(A,B),C) or as \texttt{MultiplyMatrices}(A,\texttt{MultiplyMatrices}(B,C)).

(2p) a) Suppose you are given the task to compute the product \( M = ABCD \) of the four matrices \( A, B, C \) and \( D \) with the dimensions \( 50 \times 10, 10 \times 40, 40 \times 30 \) and \( 30 \times 5 \) respectively. In how many different ways can \( M \) be evaluated? Which order of evaluating \( M \) is the best in terms of the total number of scalar multiplications required for evaluating \( M \), using the program above? (justify your answer)

(1p) b) Is the order you found in (5.a) always the best one, regardless of the dimensions of the matrices? (justify your answer)

(4p) c) Consider the general problem of finding the best order in which to evaluate the product \( A_0A_1 \cdots A_n \) for any sequence of matrices \( A_0, A_1, \ldots, A_n \). Assume that the sequence is represented as an array of links to the matrices illustrated in the figure below.

Construct an algorithm, \texttt{FindMin}(\texttt{Left}, \texttt{Right}), where the arguments \texttt{Left} and \texttt{Right} are indices as illustrated, and which computes the number of scalar multiplications required for evaluating the product \( A_{\texttt{Left}}A_{\texttt{Left}+1} \cdots A_{\text{\texttt{Right}-1}}A_{\texttt{Right}} \) in the best possible order. The algorithm need not return the actual order itself.

You have access to a procedure \texttt{Complexity}(\texttt{LX}, \texttt{RX}, \texttt{LY}, \texttt{RY}) taking four arguments which are indices to the array, such that \( L_X < R_X \) and \( L_Y < R_Y \) (otherwise it returns NULL). \texttt{Complexity}(\texttt{LX}, \texttt{RX}, \texttt{LY}, \texttt{RY}) returns the number of scalar multiplications required to evaluate the product \( Z = XY \) of the two matrices \( X \) and \( Y \) where \( X = A_{L_X}A_{L_X+1} \cdots A_{R_X-1}A_{R_X} \) and \( Y = A_{L_Y}A_{L_Y+1} \cdots A_{R_Y-1}A_{R_Y} \). (It is implemented simply so as to compute the value \( pqr \), where \( p \) is the number of rows in \( A_{L_X} \), \( q \) is the number of columns in \( A_{R_X} \) and \( r \) is the number of columns in \( A_{R_Y} \). It also executes a test to verify that the number of columns in \( A_{R_X} \) equals the number of rows in \( A_{L_Y} \). Otherwise the matrix product is not defined, and in this case \texttt{Complexity} returns NULL.)

Hint: Use divide-and-conquer.
6. (4p) In the Steiner tree problem in an undirected graph $G = (N, B)$ one shall find the minimal (cheapest) tree that spans the nodes in $N'$, where $N' \subset N$. The tree must contain all nodes in $N'$, but may also contain nodes in $N \setminus N'$ if this decreases the cost.

(2p) a) Describe a heuristic that starts with finding a minimal spanning tree with a standard method, and then tries to improve the solution. (Hint: There are similarities with certain heuristics for the traveling salesman problem) Make sure that method is polynomial and show it.

(2p) b) Apply the heuristic to the example below, where $N' = \{1, 3, 4\}$. The cost for the obtained solution in this example must be at least 2 units lower than the cost for the minimal spanning tree.

```
1 2 3 4 5 6
6 2 4 5 4 8
5 5
```

7. (4p)

(1.5p) a) Insert the numbers 3, 36, 7, 12, 13, 4, 23, 9, 22 (in this order) in a hash table of size 10, using open addressing with linear probing and the hash function $h(k) = k \mod 10$.

(0.5p) b) How many probe operations are now required for findElement(22)?

(1p) c) Explain what is meant by 'load factor' and its relation to the (expected) running time of dictionary operations. Explain what is meant by 'rehashing'.

(1p) d) For open addressing, the operation removeElement can be a bit complicated. Why? Describe a common way to approach this, and how it affects the dictionary operations findElement and insertItem.
8. (3p) Consider the directed network given by the following table. For each arc (row), \(i\) is the starting node, \(j\) the ending node, \(c_{ij}\) the arc cost, \(u_{ij}\) the capacity and \(x_{ij}\) the flow. (All lower bounds are zero.)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>(c_{ij})</th>
<th>(u_{ij})</th>
<th>(x_{ij})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>5</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>5</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Find the maximal flow from node 1 to node 5. Start from the flow given above. Give minimum cut. Give the complexity for the method used.

9. (3p) Given the following binary search tree:

(0.5p) a) Is this tree a proper binary tree? (justify your answer)

(0.5p) b) Is this tree a complete binary tree? (justify your answer)

(0.5p) c) Is this tree an AVL tree? (justify your answer)

(0.5p) d) Is the subtree with root 35 an AVL tree? (justify your answer)

(0.5p) e) Give the sequences of keys resulting from a preorder, inorder and postorder traversal of the tree.

(0.5p) f) Show the tree resulting from a (single) left rotation at the root node. Is the resulting tree an AVL tree? (justify your answer)