Tutorial 1: Exercises



27. A production manager is interested in the mean weight of items turned out by a particular process. He feels that the weight of items from the process is normally distributed with mean $\tilde{\mu}$ and that $\tilde{\mu}$ is either 109.4, 109.7, 110.0, 110.3, or 110.6. The production manager assesses prior probabilities of $P(\tilde{\mu} = 109.4) = 0.05$, $P(\tilde{\mu} = 109.7) = 0.20$, $P(\tilde{\mu} = 110.0) = 0.50$, $P(\tilde{\mu} = 110.3) = 0.20$, and $P(\tilde{\mu} = 110.6) = 0.05$. From past experience, he is willing to assume that the process variance is $\sigma^2 = 4$. He randomly selects five items from the process and weighs them, with the following results: 108, 109, 107.4, 109.6, and 112. Find the production manager's posterior distributions.



Data: $\mathbf{y} = \{108.0 \ 109.0 \ 107.4 \ 109.6 \ 112.0\} \sim N(\tilde{\mu}, \sigma^2 \approx 4)$

Sample point density: $f(y|\tilde{\mu} = \mu) = (2\pi\sigma^2)^{-0.5}e^{-\frac{(y-\mu)^2}{2\sigma^2}}$; $\sigma^2 = 4$

Likelihood:
$$L(\mu; \mathbf{y}) = \prod_{j=1}^{n=5} f(y_j | \tilde{\mu} = \mu) = \prod_{j=1}^{n=5} (2\pi\sigma^2)^{-0.5} e^{-\frac{(y_j - \mu)^2}{2\sigma^2}}$$

$$= (2\pi\sigma^2)^{-0.5n} e^{-\frac{1}{2\sigma^2}\sum_{j=1}^n (y_j - \mu)^2} = (8\pi)^{-2.5} e^{-\frac{1}{8}\sum_{j=1}^5 (y_j - \mu)^2}$$

$$= (8\pi)^{-2.5}e^{-\frac{1}{8}\left[(108-\mu)^2 + (109-\mu)^2 + (107.4-\mu)^2 + (109.6-\mu)^2 + (112-\mu)^2\right]}$$

Posterior distribution of $\tilde{\mu}$:

$$\begin{split} f''(\mu|\mathbf{y}) &= \frac{L(\mu;\mathbf{y}) \cdot f'(\mu)}{\sum_{i=1}^{5} L(\mu_i;\mathbf{y}) \cdot f'(\mu_i)} = \frac{(2\pi\sigma^2)^{-0.5n}e^{-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(y_j-\mu)^2} \cdot f'(\mu)}{\sum_{i=1}^{5} (2\pi\sigma^2)^{-0.5n}e^{-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(y_j-\mu_i)^2} \cdot f'(\mu_i)} \\ &= \frac{e^{-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(y_j-\mu)^2} \cdot f'(\mu)}{\sum_{i=1}^{5} e^{-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(y_j-\mu_i)^2} \cdot f'(\mu_i)} = \\ &= \frac{e^{-\frac{1}{8}\sum_{j=1}^{n}(y_j-\mu)^2} \cdot f'(\mu_i)}{e^{-\frac{1}{8}[(108-109.4)^2+(109-109.4)^2+(107.4-109.4)^2+(109.6-109.4)^2+(112-109.4)^2]} \cdot 0.05 + \\ &e^{-\frac{1}{8}[(108-109.4)^2+(109-109.7)^2+(107.4-109.7)^2+(109.6-109.7)^2+(112-109.7)^2]} \cdot 0.20 + \\ &e^{-\frac{1}{8}[(108-110.0)^2+(109-110.0)^2+(107.4-110.0)^2+(109.6-110.0)^2+(112-110.3)^2]} \cdot 0.20 + \\ &e^{-\frac{1}{8}[(108-110.3)^2+(109-110.3)^2+(107.4-110.6)^2+(109.6-110.6)^2+(112-110.6)^2]} \cdot 0.20 + \\ &e^{-\frac{1}{8}[(108-110.6)^2+(109-110.6)^2+(107.4-110.6)^2+(109.6-110.6)^2+(112-110.6)^2]} \cdot 0.05 \\ &= \frac{e^{-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(y_j-\mu)^2} \cdot f'(\mu)}{e^{-\frac{1}{8}[12.92} \cdot 0.05 + e^{-\frac{1}{8}(13.97} \cdot 0.20 + e^{-\frac{1}{8}(15.92} \cdot 0.50 + e^{-\frac{1}{8}(18.77} \cdot 0.20 + e^{-\frac{1}{8}(22.52} \cdot 0.05)} \\ &\approx \frac{e^{-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(y_j-\mu)^2} \cdot f'(\mu)}{e^{-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(y_j-\mu)^2} \cdot f'(\mu)} \end{split}$$

0.1353

$$\begin{aligned} f''(\mu_{1}|\mathbf{y}) &= \underline{f''(109.4|\mathbf{y})}_{0.1353} \approx \frac{e^{-\frac{1}{8}[(108-109.4)^{2}+(109-109.4)^{2}+(107.4-109.4)^{2}+(109.6-109.4)^{2}+(112-109.4)^{2}]_{\cdot 0.05}}{0.1353} \approx \\ \underline{0.0735}\\ f''(\mu_{2}|\mathbf{y}) &= \underline{f''(109.7|\mathbf{y})}_{0.2578} \approx \frac{e^{-\frac{1}{8}[(108-109.7)^{2}+(109-109.7)^{2}+(107.4-109.7)^{2}+(109.6-109.7)^{2}+(112-109.7)^{2}]_{\cdot 0.20}}{0.1353} \approx \\ \underline{0.2578}\\ f''(\mu_{3}|\mathbf{y}) &= \underline{f''(110.0|\mathbf{y})}_{0.5051} \approx \frac{e^{-\frac{1}{8}[(108-110.0)^{2}+(109-110.0)^{2}+(107.4-110.0)^{2}+(109.6-110.0)^{2}+(112-110.0)^{2}]_{\cdot 0.50}}{0.1353} \approx \\ \underline{0.5051}\\ f''(\mu_{4}|\mathbf{y}) &= \underline{f''(110.3|\mathbf{y})}_{0.1415} \approx \frac{e^{-\frac{1}{8}[(108-110.3)^{2}+(109-110.3)^{2}+(107.4-110.3)^{2}+(109.6-110.3)^{2}+(112-110.3)^{2}]_{\cdot 0.20}}{0.1353} \approx \\ \underline{0.0221} &= \underline{f''(110.6|\mathbf{y})}_{0.1353} \approx \frac{e^{\frac{1}{8}[(108-110.6)^{2}+(109-110.6)^{2}+(107.4-110.6)^{2}+(109.6-110.6)^{2}+(112-110.6)^{2}]_{\cdot 0.05}}}{0.1353} \approx \\ \end{array}$$



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$$\underline{E_{\text{prior}}(\tilde{\mu})} = E(\tilde{\mu}) = 109.4 \cdot 0.05 + 109.7 \cdot 0.20 + 110.0 \cdot 0.50 \\
 +110.3 \cdot 0.20 + 110.6 \cdot 0.05 = \underline{110} \quad \text{(obvious?)}$$

$$\underline{Var_{\text{prior}}(\tilde{\mu})} = Var(\tilde{\mu}) = E(\tilde{\mu}^2) - (E(\tilde{\mu}))^2 \\
 = 109.4^2 \cdot 0.05 + 109.7^2 \cdot 0.20 + 110.0^2 \cdot 0.50 \\
 +110.3^2 \cdot 0.20 + 110.6^2 \cdot 0.05 - 110^2 = \underline{0.072}$$

$$\underline{E_{\text{posterior}}(\tilde{\mu})} = E(\tilde{\mu}|\mathbf{y}) \\
 = 109.4 \cdot 0.07348... + 109.7 \cdot 0.25780... + 110.0 \cdot 0.50508... \\
 +110.3 \cdot 0.14148 + 110.6 \cdot 0.02213 \approx 109.9$$

 $Var_{\text{posterior}}(\tilde{\mu}) = Var(\tilde{\mu}|\mathbf{y}) = E(\tilde{\mu}^2|\mathbf{y}) - (E(\tilde{\mu}|\mathbf{y}))^2$

 $\approx 109.4^2 \cdot 0.07348... + 109.7^2 \cdot 0.25780... + 110.0^2 \cdot 0.50508... + 110.3^2 \cdot 0.14148... + 110.6^2 \cdot 0.02213... - 109.7^2 \approx 0.066$

In Exercise 27, if $\tilde{\mu}$ is assumed to be continuous and if the prior distribution for $\tilde{\mu}$ is a normal distribution with mean 110 and variance 0.4, find the posterior distribution.

Prior distribution: $\tilde{\mu} \sim N(110, 0.4) = N(m', \sigma'^2)$

Prior density: $p(\mu) = f'(\mu) = (2\pi\sigma'^2)^{-0.5}e^{-\frac{(\mu-m')^2}{2\sigma'^2}} = (2\pi \cdot 0.4)^{-0.5}e^{-\frac{(\mu-110)^2}{0.8}}$

Data: $\mathbf{y} = \{108.0 \ 109.0 \ 107.4 \ 109.6 \ 112.0\} \sim N(\tilde{\mu}, \sigma^2 \approx 4)$

$$\bar{y} = \frac{108 + 109 + 107.4 + 109.6 + 112}{5} = 109.2$$
$$s^{2} = \frac{1}{4} \sum_{j=1}^{5} (y_{j} - 109.2)^{2} = 3.18$$

$$f''^{(\mu|\mathbf{y})} = \frac{L(\mu; \mathbf{y}) \cdot f'(\mu)}{\int_{-\infty}^{\infty} L(\mu; \mathbf{y}) \cdot f'(\mu) d\mu} = \begin{pmatrix} L(\mu; \mathbf{y}) \text{ from} \\ \text{Exercise } 4.27 \end{pmatrix} =$$
$$= \frac{(2\pi\sigma^2)^{-0.5n} e^{-\frac{1}{2\sigma^2} \sum_{j=1}^{n} (y_j - \mu)^2} \cdot (2\pi\sigma'^2)^{-0.5} e^{-\frac{(\mu - m')^2}{2\sigma'^2}}}{\int_{-\infty}^{\infty} (2\pi\sigma^2)^{-0.5n} e^{-\frac{1}{2\sigma^2} \sum_{j=1}^{n} (y_j - \mu)^2} \cdot (2\pi\sigma'^2)^{-0.5} e^{-\frac{(\mu - m')^2}{2\sigma'^2}} d\mu} = \begin{pmatrix} \text{"Completion} \\ \text{of squares"} \end{pmatrix} =$$

$$= (2\pi\sigma''^2)^{-0.5} e^{-\frac{(\mu - m'')^2}{2\sigma''^2}}$$

where

$$m'' = \frac{(1/\sigma'^2) \cdot m' + (n/\sigma^2) \cdot m}{(1/\sigma'^2) + (n/\sigma^2)} = \frac{(1/\sigma'^2) \cdot m' + (n/\sigma^2) \cdot \bar{y}}{(1/\sigma'^2) + (n/\sigma^2)}$$
$$= \frac{(1/0.4) \cdot 110 + (5/4) \cdot 109.2}{(1/0.4) + (5/4)} \approx 109.7$$

$$\sigma''^{2} = \frac{\sigma^{2} \cdot \sigma'^{2}}{\sigma^{2} + n \cdot \sigma'^{2}} = \frac{4 \cdot 0.4}{4 + 5 \cdot 0.4} \approx 0.267$$

Thus, the posterior distribution is $N(m''=109.7, \sigma''^2=0.267)$

A particular product is both manufactured and marketed by two different firms. The total demand for the product is virtually fixed, so neither firm has advertised in the past. However, the owner of Firm A is considering an advertising campaign to woo customers away from Firm B. The ad campaign she has in mind will cost \$200,000. She is uncertain about the number of customers that will switch to her firm as a result of the advertising, but she is willing to assume that she will gain either 10 percent, 20 percent, or 30 percent of the market. For each 10 percent gain in market share, the firm's profits will increase by \$150,000. Construct the payoff table for this problem and find the corresponding loss table.

Components of the decision problem:

Actions (equal procedures here): Run the campaign (a_1) or not (a_2) . States of the world: Gain 10 percent (θ_1) , 20 percent (θ_2) or 30 percent (θ_3) . Consequences: Payoffs: with a_1 : 150000 $\cdot(\theta/10) - 200000$, with a_2 : 0

Note! One could consider adding a fourth state of the world: "gain 0%". This will be the only state possible with action a_2 but impossible with action a_1 . However, since with this state the maximum payoff (and hence loss) will be 0 it will have no effect on the decision problem.

Payoff table

	$\theta = 10\%$	$\theta = 20\%$	$\theta = 30\%$
Run campaign (a_1)	-50000	100000	250000
<i>Do not run campaign</i> (a_2)	0	0	0

$$\begin{aligned} L_{ij} &= \max_{k} R_{kj} - R_{ij} \\ \Rightarrow L_{i1} &= 0 - R_{i1} \quad ; \quad L_{i2} = 100000 - R_{i2} \quad ; \quad L_{i3} = 250000 - R_{i2} \end{aligned}$$

 \Rightarrow Loss table

	heta=10%	$\theta = 20\%$	$\theta = 30\%$
Run campaign (a_1)	50000	0	0
<i>Do not run campaign</i> (a_2)	0	100000	250000

In Exercise 6, the owner of Firm A is worried that if she proceeds with the ad campaign, Firm B will do likewise, in which case the market shares of the two firms will remain constant. How does this affect the set of possible states of the world, *S*? Construct a modified payoff table to allow for this change.

Firm B is assumed to run a campaign <u>only</u> if Firm A does so. \Rightarrow Firm A cannot <u>lose</u> any market shares, but the gain of new market may be zero \Rightarrow One new state of the world: "gain is 0%"

Modified payoff table

	$\theta = 0\%$	$\theta = 10\%$	$\theta = 20\%$	$\theta = 30\%$
Run campaign (a_1)	-200000	-50000	100000	250000
Do not run campaign (a_2)	0	0	0	0

In Exercise 6, the owner of Firm A feels that if the ad campaign is initiated, the events "gain 20 percent of the market" and "gain 30 percent of the market" are equally likely and each of these events is three times as likely as the event "gain 10 percent of the market." Using the *ER* criterion, what should she do?

Maximise expected payoff

The payoff table is

	$\theta = 10\%$	$\theta = 20\%$	$\theta = 30\%$
Run campaign (a_1)	-50000	100000	250000
<i>Do not run campaign</i> (a_2)	0	0	0

$$P(\tilde{\theta} = 20\%) = P(\tilde{\theta} = 30\%) = 3 \cdot P(\tilde{\theta} = 10\%)$$

$$P(\tilde{\theta} = 20\%) + P(\tilde{\theta} = 30\%) + P(\tilde{\theta} = 10\%) = 1$$

$$\Rightarrow P(\tilde{\theta} = 10\%) = 1/7 \text{ and } P(\tilde{\theta} = 20\%) = P(\tilde{\theta} = 30\%) = 3/7$$

Hence, $E(R(a_1, \theta)) = (-50000) \cdot \frac{1}{7} + 100000 \cdot \frac{3}{7} + 250000 \cdot \frac{3}{7} = \frac{1000000}{7} \approx 143000$ $ER((a_2, \theta)) = 0$ To formally show this, we could have introduced the state " $\theta = 0\%$ " and put a point mass of probability 1

on this state when the decision is a_2

 \Rightarrow Run campaign!

In Exercise 17, suppose that the probabilities given are conditional on the rival firm not advertising. If Firm B also advertises, then the owner of Firm A is certain (for all practical purposes) that there will be no change in the market share of either firm. She thinks that the chances are 2 in 3 that Firm B will advertise if Firm A does. What should the owner of Firm A do?

$$P(\tilde{\theta} = 10\%|B \text{ not advertising}) = 1/7 \text{ and}$$

 $P(\tilde{\theta} = 20\%|B \text{ not advertising}) = P(\tilde{\theta} = 30\%|B \text{ not advertising}) = 3/7$
 $P(\tilde{\theta} = 0\%|B \text{ advertising}) = 1$
 $P(B \text{ advertising}|A \text{ advertising}) = 2/3$

 $\Rightarrow P(\tilde{\theta} = 10\%) = P(\tilde{\theta} = 10\%|B \text{ not advertising}) \cdot P(B \text{ not advertising})$ $+ P(\tilde{\theta} = 10\%|B \text{ advertising}) \cdot P(B \text{ advertising}) = (1/7) \cdot (1/3) + 0 \cdot (2/3) = 1/21$ $P(\tilde{\theta} = 20\%) = P(\tilde{\theta} = 20\%|B \text{ not advertising}) \cdot P(B \text{ not advertising})$ $+ P(\tilde{\theta} = 20\%|B \text{ advertising}) \cdot P(B \text{ advertising}) = (3/7) \cdot (1/3) + 0 \cdot (2/3) = 3/21$ $P(\tilde{\theta} = 30\%) = P(\tilde{\theta} = 20\%) = 3/21$ $P(\tilde{\theta} = 0\%) = P(\tilde{\theta} = 0\%|B \text{ not advertising}) \cdot P(B \text{ not advertising})$ $+ P(\tilde{\theta} = 0\%|B \text{ advertising}) \cdot P(B \text{ advertising}) = 0 \cdot (1/3) + 1 \cdot (2/3) = 2/3$

The payoff table applicable is

	$\theta = 0\%$	$\theta = 10\%$	$\theta = 20\%$	$\theta = 30\%$
Run campaign (a_1)	-200000	-50000	100000	250000
Do not run campaign (a_2)	0	0	0	0

Hence,

$$\begin{aligned} & ER(a_1) = (-200000) \cdot \frac{2}{3} + (-50000) \cdot \frac{1}{21} + 100000 \cdot \frac{3}{21} + 250000 \cdot \frac{3}{21} \\ &= -\frac{1800000}{7} \approx -86000 \\ & ER(a_2) = 0 \end{aligned}$$

 \Rightarrow Do not run campaign!