Lecture 8 (*not given in class*) : More utility theory and exercises

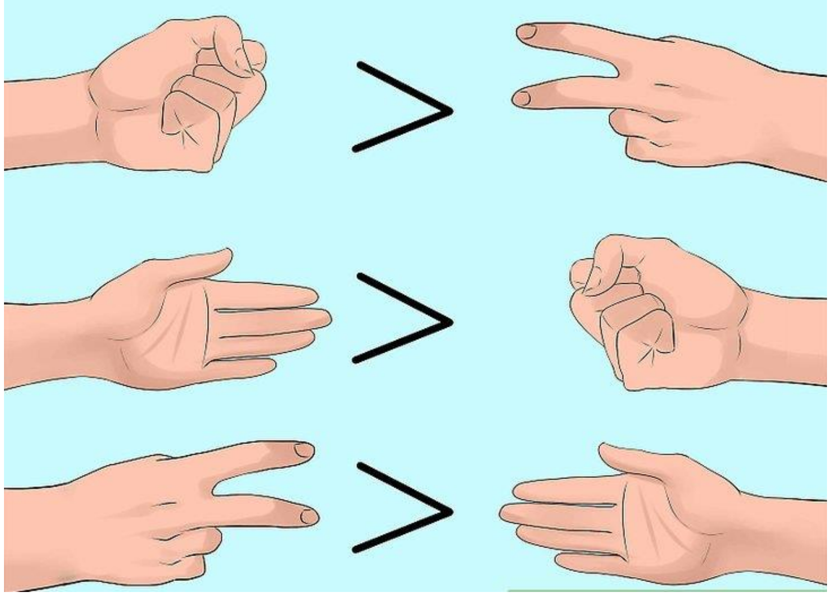
Axioms of coherence for a utility function (von Neumann & Morgenstern, 1947, Theory of Games and Economic Behaviour, 2nd ed., Princeton University Press)

1. Ordering of consequences: It is possible for the decision-maker to order the possible outcomes from best to worst (or to explicitly state their indifference between two or several of them)
2. Transitivity of preferences: If the relative preferences of three possible outcomes, expressed as utilities U_1, U_2 and U_3, are such that $U_2 > U_1$ and $U_3 > U_2$, then U_3 must be greater than U_1, i.e. $U_3 > U_1$
3. Continuity of preferences: If $U_3 > U_2 > U_1$ then it is possible to find a *p-mixture* of U_1 and U_3 that is preferable to (>) U_2 and another *p*-mixture of U_1 and U_3 such that U_2 is preferred to (>) that *p*-mixture
4. Independence: If $U_2 > U_1$ then for any another utility U_3 it holds that a *p*-mixture of U_2 and U_3 is preferred to the "same" *p*-mixture of U_1 and U_3, i.e. $pU_2 + (1-p)U_3 > pU_1 + (1-p)U_3$

<u>Transitivity of preferences:</u> If the relative preferences of three possible outcomes, expressed as utilities U_1, U_2 and U_3, are such that $U_2 > U_1$ and $U_3 > U_2$, then U_3 must be greater than U_1, i.e. $U_3 > U_1$

How would relative preferences be if the transitivity axiom <u>is not</u> satisfied?

Example: Rock, Paper, Scissors



Exercise 5.36

For each of the following utility functions for changes in assets (monetary payoffs), graph the function and comment on the attitude toward risk that is implied by the function. All of the functions are defined for $-1000 < R < 1000$.

(a) $U(R) = (R+1000)^2$.

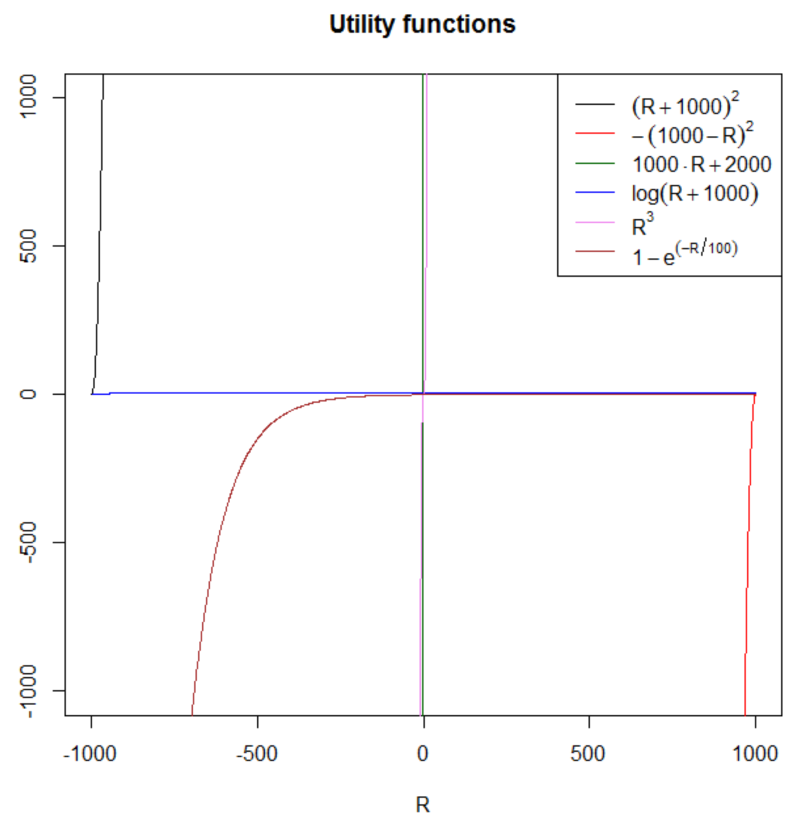
(b) $U(R) = -(1000-R)^2$.

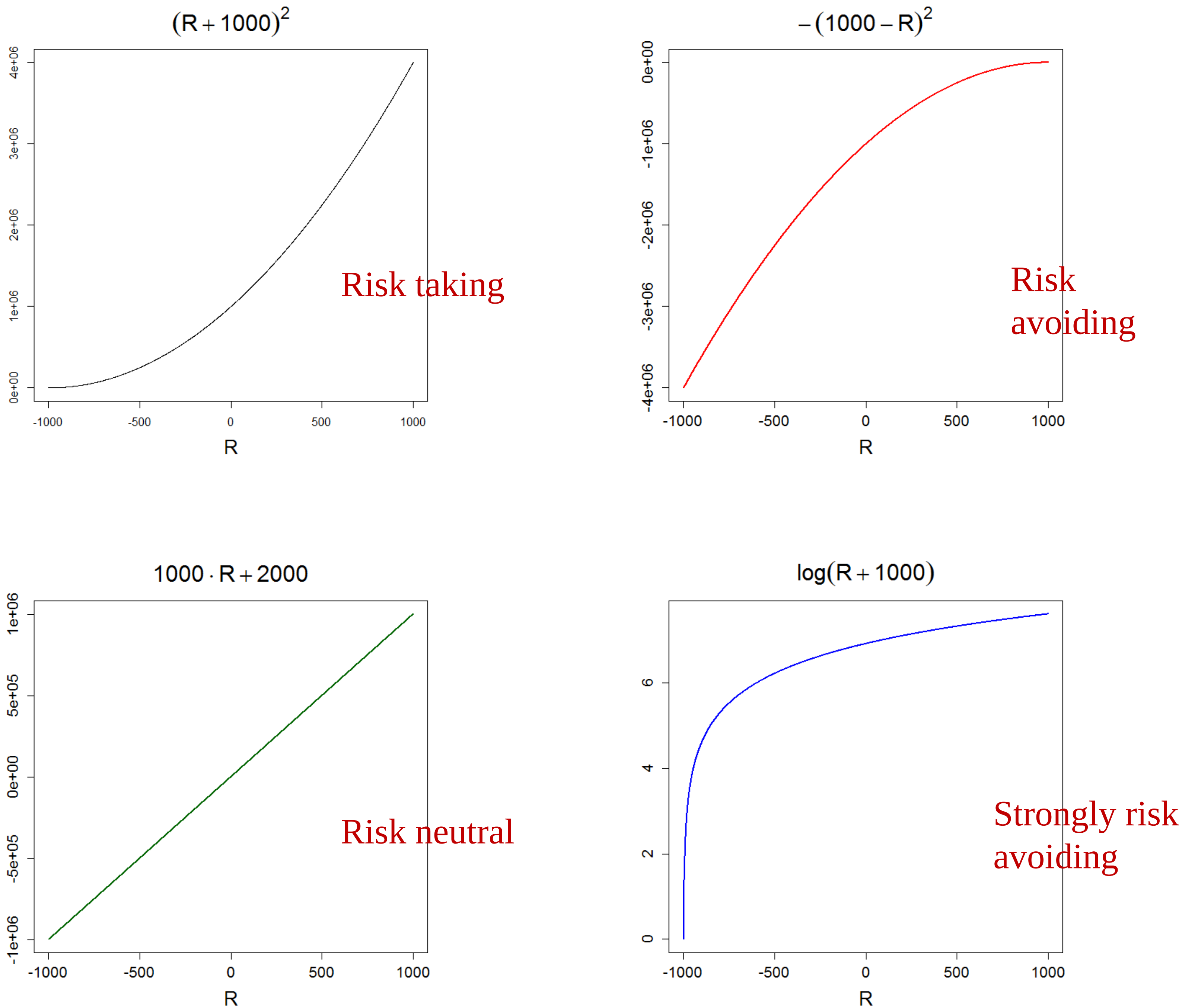
(c) $U(R) = 1000R+2000$.

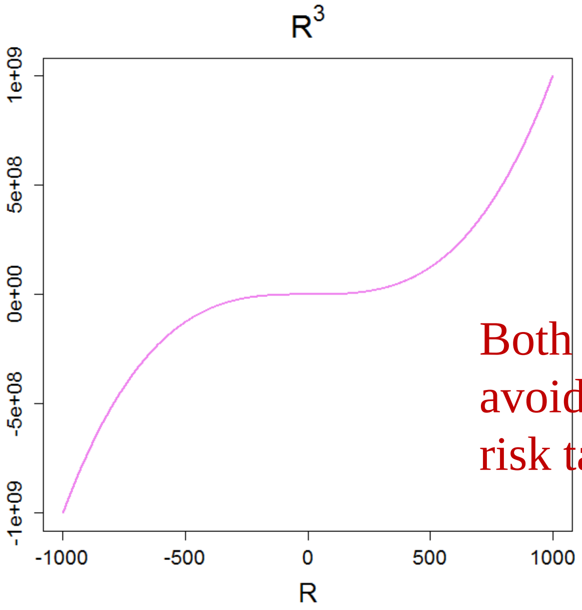
(d) $U(R) = \log(R+1000)$.

(e) $U(R) = R^3$.

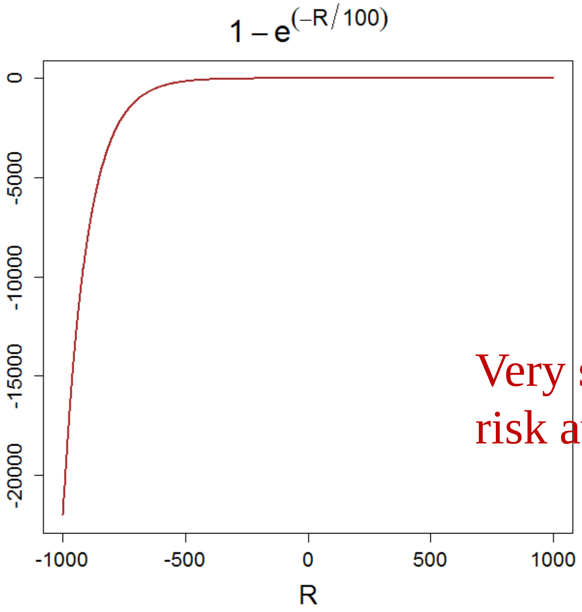
(f) $U(R) = 1-e^{-R/100}$.




$(R + 1000)^2$
Risk taking
R
$-(1000 - R)^2$
Risk avoiding
R
$1000 \cdot R + 2000$
Risk neutral
R
$\log(R + 1000)$
Strongly risk avoiding
R



Both risk avoiding and risk taking



Very strongly risk avoiding

Exercise 5.37

For each of the utility functions in Exercise 36, find out if the decision maker should take a bet in which he will win \$100 with probability p and lose \$50 with probability $1 - p$,
(a) if $p = 1/2$,
(b) if $p = 1/3$,
(c) if $p = 1/4$.

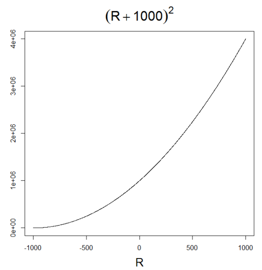
$$EU = U(100) \cdot p + U(-50) \cdot (1 - p)$$

(a) $EU = U(100) \cdot 0.5 + U(-50) \cdot 0.5$

$ER = 100 \cdot 0.5 + (-50) \cdot 0.5 = 25$

<u>5.36 (a):</u>

$U(R) = (R + 1000)^2 \Rightarrow EU = 1100^2 \cdot 0.5 + 950^2 \cdot 0.5 = 1056250$



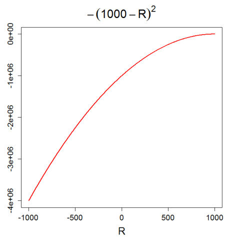
$R \geq -1000$

$CE = U^{-1}(EU) = \sqrt{EU} - 1000 = \sqrt{1056250} - 1000 = 27.74 > ER$

$\Rightarrow$ Take the bet!

5.36 (b):

$$U(R) = -(1000 - R)^2 \Rightarrow EU = -900^2 \cdot 0.5 + (-1050^2) \cdot 0.5 = -956250$$

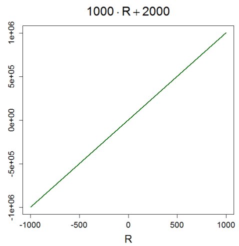


$$CE = U^{-1}(EU) \overset{R \le 1000}{=} 1000 - \sqrt{-EU} = 1000 - \sqrt{-(-956250)} = 22.12 < 25 = ER$$

$\Rightarrow$ Do not take the bet!

5.36 (c):

$$U(R) = 1000 \cdot R + 2000 \Rightarrow EU = 102000 \cdot 0.5 + (-48000) \cdot 0.5 = 27000$$

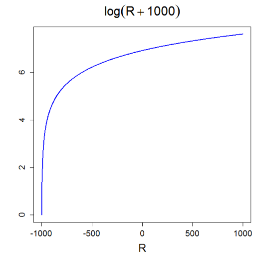


$$CE = U^{-1}(EU) = (EU - 2000)/1000 = 25000/1000 = 25 = ER$$

$\Rightarrow$ Indifferent!

5.36 (d):

$U(R) = \log(R + 1000) \Rightarrow EU = \log(1100) \cdot 0.5 + \log(950) \cdot 0.5 = 6.93$

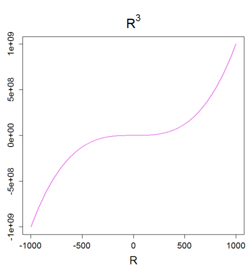


$CE = U^{-1}(EU) = \exp(EU) – 1000 = \exp(\log(1100) \cdot 0.5 + \log(950) \cdot 0.5)$
$– 1000 = 22.25 < 25 = ER$

$\Rightarrow$ Do not take the bet!

5.36 (e):

$U(R) = R^3 \Rightarrow EU = 100^3 \cdot 0.5 + (–50)^3 \cdot 0.5 = 437500$

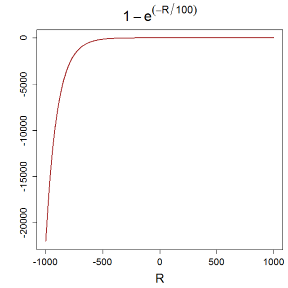


$CE = U^{-1}(EU) = (EU)^{1/3} = (437500)^{1/3} = 75.91 > 25 = ER$

$\Rightarrow$ Take the bet!

5.36 (f):

$U(R) = 1 - \exp(-R/100) \Rightarrow EU = (1 - \exp(-1)) \cdot 0.5 + (1 - \exp(0.5)) \cdot 0.5 =$
$-\ 0.0083$



$CE = U^{-1}(EU) = -100 \cdot \log(1 - EU) =$
$-100 \cdot \log(1 - ((1 - \exp(-1)) \cdot 0.5 + (1 - \exp(0.5)) \cdot 0.5)) = -0.83 < 25 = ER$

$\Rightarrow$ Do not take the bet!

(b) $EU = U(100) \cdot (1/3) + U(-50) \cdot (2/3)$

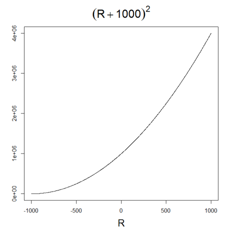
$ER = 100 \cdot (1/3) + (-50) \cdot (2/3) = 0$

<u>5.36 (a):</u>

$U(R) = (R + 1000)^2 \Rightarrow EU = 1100^2 \cdot (1/3) + 950^2 \cdot (2/3) = 1005000$

$CE = U^{-1}(EU) = \sqrt{EU} - 1000 = \sqrt{1005000} - 1000 = 2.50 > ER$

$\Rightarrow$ Take the bet!

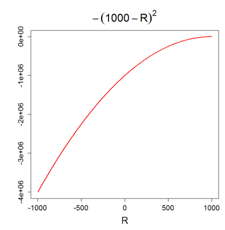


<u>5.36 (b):</u>

$U(R) = -(1000 - R)^2 \Rightarrow EU = -900^2 \cdot (1/3) + (-1050^2) \cdot (2/3) = -1005000$

$CE = U^{-1}(EU) = 1000 - \sqrt{-EU} = 1000 - \sqrt{-(-1005000)} =$
$-2.50 < 0 = ER$

$\Rightarrow$ Do not take the bet!

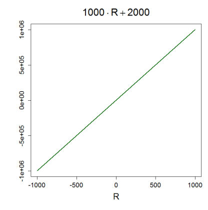


<u>5.36 (c):</u>

$U(R) = 1000 \cdot R + 2000 \Rightarrow EU = 102000 \cdot (1/3) + (-48000) \cdot (2/3) = 2000$

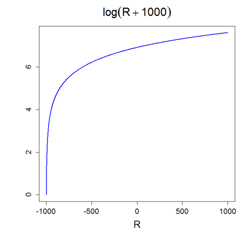
$CE = U^{-1}(EU) = (EU - 2000)/1000 = 0/1000 = 0 = ER$

$\Rightarrow$ Indifferent! (Expected?)



5.36 (d):

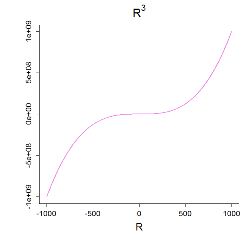
$U(R) = \log(R + 1000) \Rightarrow EU = \log(1100) \cdot (1/3) + \log(950) \cdot (2/3) = 6.91$



$CE = U^{-1}(EU) = \exp(EU) - 1000 =$
$\exp(\log(1100) \cdot (1/3) + \log(950) \cdot (2/3)) - 1000 = -2.42 < 0 = ER$

$\Rightarrow$ Do not take the bet!

5.36 (e):



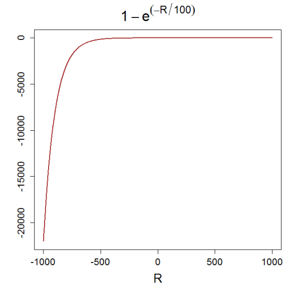
$U(R) = R^3 \Rightarrow EU = 100^3 \cdot (1/3) + (-50)^3 \cdot (2/3) = 250000$

$CE = U^{-1}(EU) = (EU)^{1/3} = (250000)^{1/3} = 63.00 > 0 = ER$

$\Rightarrow$ Take the bet!

5.36 (f):

$U(R) = 1 - \exp(-R/100) \Rightarrow EU = (1 - \exp(-1)) \cdot (1/3) + (1 - \exp(0.5)) \cdot (2/3) =$
-0.22



$CE = U^{-1}(EU) = -100 \cdot \log(1 - EU) =$
$-100 \cdot \log(1 - ((1 - \exp(-1)) \cdot (1/3) + (1 - \exp(0.5)) \cdot (2/3)))$
$= -20.03 < 0 = ER$

$\Rightarrow$ Do not take the bet!

(c) $EU = U(100) \cdot (1/3) + U(-50) \cdot (2/3)$

$ER = 100 \cdot (1/4) + (-50) \cdot (3/4) = -12.5$

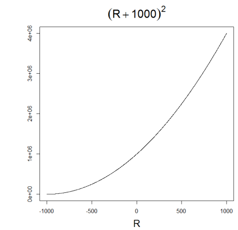
<u>5.36 (a):</u>

$U(R) = (R + 1000)^2 \Rightarrow EU = 1100^2 \cdot (1/4) + 950^2 \cdot (3/4) = 979375$

$CE = U^{-1}(EU) = \sqrt{EU} - 1000 = \sqrt{979375} - 1000 = -10.37 > ER$

but still negative!

$\Rightarrow$ Do not take the bet!



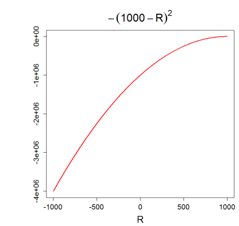
<u>5.36 (b):</u>

$U(R) = -(1000 - R)^2 \Rightarrow EU = -900^2 \cdot (1/4) + (-1050^2) \cdot (3/4) = -1029375$

$CE = U^{-1}(EU) = 1000 - \sqrt{-EU} = 1000 - \sqrt{-(-1029375)} =$

$-14.58 < -12.5 = ER < 0$

$\Rightarrow$ Do not take the bet!

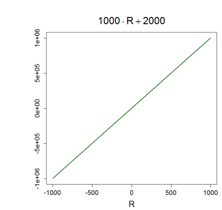


<u>5.36 (c):</u>

$U(R) = 1000 \cdot R + 2000 \Rightarrow EU = 102000 \cdot (1/4) + (-48000) \cdot (3/4) = -10500$

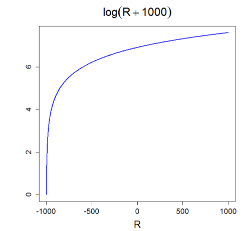
$CE = U^{-1}(EU) = (EU - 2000)/1000 = -12.5 = ER < 0$

$\Rightarrow$ Indifferent in terms of utility, but with $CE < 0$ do not take the bet!



5.36 (d):

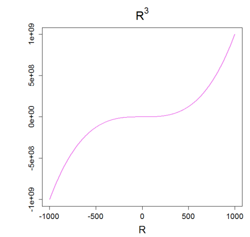
$U(R) = \log(R + 1000) \Rightarrow EU = \log(1100) \cdot (1/4) + \log(950) \cdot (3/4) = 6.89$



$CE = U^{-1}(EU) = \exp(EU) - 1000 =$
$\exp(\log(1100) \cdot (1/4) + \log(950) \cdot (3/4)) - 1000 = -14.54 < -12.5$
$= ER < 0$

$\Rightarrow$ Do not take the bet!

5.36 (e):



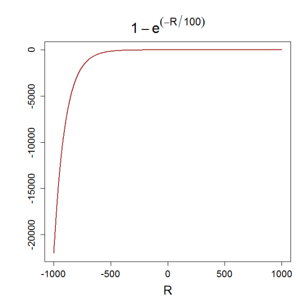
$U(R) = R^3 \Rightarrow EU = 100^3 \cdot (1/4) + (-50)^3 \cdot (3/4) = 156250$

$CE = U^{-1}(EU) = (EU)^{1/3} = (156250)^{1/3} = 53.86 > 0 > ER = -12.5$

$\Rightarrow$ Take the bet!

5.36 (f):

$U(R) = 1 - \exp(-R/100) \Rightarrow EU = (1 - \exp(-1)) \cdot (1/4) + (1 - \exp(0.5)) \cdot (3/4) =$
-0.33



$CE = U^{-1}(EU) = -100 \cdot \log(1 - EU) =$
$-100 \cdot \log(1 - ((1 - \exp(-1)) \cdot (1/4) + (1 - \exp(0.5)) \cdot (3/4)))$
$= -28.41 < -12.5 = ER < 0$

$\Rightarrow$ Do not take the bet!

Exercise 5.45

For each of the utility functions in Exercise 36, find the risk premiums for the following gambles.

(a) You win \$100 with probability 0.5 and you lose \$100 with probability 0.5.
(b) You win \$100 with probability 0.4 and you lose \$50 with probability 0.6.
(c) You win \$70 with probability 0.3 and you lose \$30 with probability 0.7.
(d) You win \$200 with probability 0.5 and you win \$50 with probability 0.5.

(a) $EU = U(100) \cdot 0.5 + U(-100) \cdot 0.5$

$ER = 100 \cdot 0.5 + (-100) \cdot 0.5 = 0$

$RP = ER - CE = -CE$

5.36 (a):

$U(R) = (R + 1000)^2 \Rightarrow EU = 1100^2 \cdot 0.5 + 900^2 \cdot 0.5 = 1010000$

$RP = -CE = -U^{-1}(EU) = -(\sqrt{EU} - 1000) = -(\sqrt{1010000} - 1000) = -5.00$

5.36 (b):

$U(R) = -(1000 - R)^2 \Rightarrow EU = -900^2 \cdot 0.5 + (-1100^2) \cdot 0.5 = -1010000$

$RP = -CE = -U^{-1}(EU) = -(1000 - \sqrt{-EU}) = -(1000 - \sqrt{-(-1010000)}) = 5.00$

5.36 (c):

$U(R\ ) = 1000 \cdot R + 2000 \Rightarrow EU = 102000 \cdot 0.5 + (-98000\ ) \cdot 0.5 = 2000$

$RP = -CE = -U^{-1}(EU) = -((EU - 2000)\ /1000) = 0$

5.36 (d):

$U(R\ ) = \log(R + 1000) \Rightarrow EU = \log(1100) \cdot 0.5 + \log(900) \cdot 0.5 = 6.90$

$RP = -CE = -U^{-1}(EU) = -(\exp(EU\ ) - 1000)\ =$
$-(\exp(\ \log(1100) \cdot 0.5 + \log(900) \cdot 0.5\ ) - 1000) = 5.01$

5.36 (e):

$U(R\ ) = R^{3} \Rightarrow EU = 100^{3} \cdot 0.5 + (-100)^{3} \cdot 0.5 = 0$

$RP = -CE = -U^{-1}(EU) = -(EU\ )^{1/3}\ = -(0)^{1/3} = 0$

5.36 (f):

$U(R\ ) = 1 - \exp(-R/100) \Rightarrow EU = (1 - \exp(-1)) \cdot 0.5 + (1 - \exp(1)) \cdot 0.5 =$
-0.54

$RP = -CE = -U^{-1}(EU) = -(-100 \cdot \log(1 - EU\ ))\ =$
$-(-100 \cdot \log(1 - ((1 - \exp(-1)) \cdot 0.5 + (1 - \exp(1)) \cdot 0.5))) = 43.38$

(b) $EU = U(100) \cdot 0.4 + U(-50) \cdot 0.6$

$ER = 100 \cdot 0.4 + (-50) \cdot 0.6 = 10$

$RP = ER - CE = 10 - CE$

<u>5.36 (a):</u>

$U(R) = (R + 1000)^2 \Rightarrow EU = 1100^2 \cdot 0.4 + 950^2 \cdot 0.6 = 1025500$

$RP = 10 - CE = 10 - U^{-1}(EU) = 10 - (\sqrt{EU} - 1000) =$

$10 - (\sqrt{1025500} - 1000) = 10 - 12.67 = -2.67$

<u>5.36 (b):</u>

$U(R) = -(1000 - R)^2 \Rightarrow EU = -900^2 \cdot 0.4 + (-1050^2) \cdot 0.6 = -985500$

$RP = 10 - CE = 10 - U^{-1}(EU) = 10 - (1000 - \sqrt{-EU})$

$= 10 - (1000 - \sqrt{-(-985500)}) = 10 - 7.28 = 2.72$

<u>5.36 (c):</u>

$U(R) = 1000 \cdot R + 2000 \Rightarrow EU = 102000 \cdot 0.4 + (-48000) \cdot 0.6 = 12000$

$RP = 10 - CE = 10 - U^{-1}(EU) = 10 - ((EU - 2000)/1000) = 10 - 10 = 0$

5.36 (d):

$U(R) = \log(R + 1000) \Rightarrow EU = \log(1100) \cdot 0.4 + \log(950) \cdot 0.6 = 6.92$

$RP = 10 - CE = 10 - U^{-1}(EU) = 10 - (\exp(EU) - 1000) =$
$10 - (\exp(\log(1100) \cdot 0.4 + \log(950) \cdot 0.6) - 1000) = 10 - 5.01 = 2.62$

5.36 (e):

$U(R) = R^3 \Rightarrow EU = 100^3 \cdot 0.4 + (-50)^3 \cdot 0.6 = 325000$

$RP = 10 - CE = 10 - U^{-1}(EU) = 10 - (EU)^{1/3} = 10 - (325000)^{1/3} = -58.75$

5.36 (f):

$U(R) = 1 - \exp(-R/100) \Rightarrow EU = (1 - \exp(-1)) \cdot 0.4 + (1 - \exp(0.5)) \cdot 0.6 =$
-0.14

$RP = 10 - CE = 10 - U^{-1}(EU) = 10 - (-100 \cdot \log(1 - EU)) =$
$10 - (-100 \cdot \log(1 - ((1 - \exp(-1)) \cdot 0.6 + (1 - \exp(0.5)) \cdot 0.4))) = 22.78$

Pratt-Arrow risk aversion function

$$r(T) = -\frac{\frac{d^2U(T)}{dT^2}}{\frac{dU(T)}{d(T)}}$$

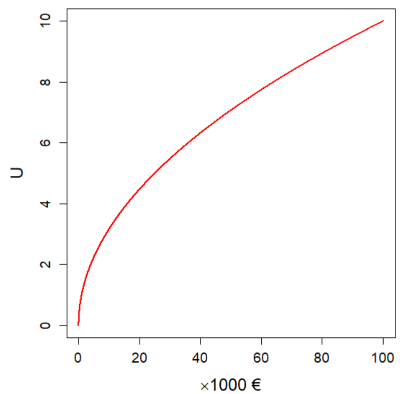
$\frac{df(x)}{dx}$ first derivative

$\frac{d^2f(x)}{dx^2}$ second derivative

measures the degree of risk aversion for a decision maker with *total assets T* (including *R* from the output of the decision problem)

Example

Assume $U(T) = \sqrt{T}$



$$\frac{dU}{dT} = \frac{1}{2\sqrt{T}}, \frac{d^2U(T)}{dT^2} = -\frac{1}{4T\sqrt{T}} \implies r(T) = -\frac{-1/4T\sqrt{T}}{1/2\sqrt{T}} = \frac{1}{2T}$$

Risk aversion decreases with the total assets

Relation between utility as a function of payoff, R and utility as a function of total assets, T:

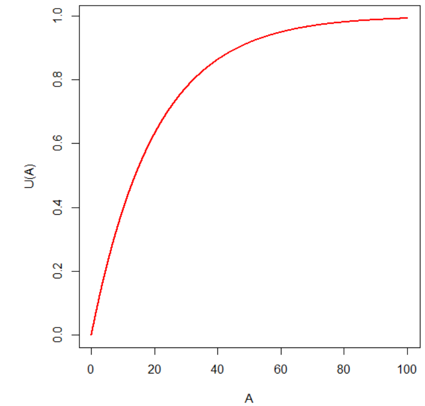
$$U_{\text{Payoff}}(R) = U_P(R) = U_{TA}(T + R) - U_{TA}(T)$$

Exercise 5.39

39. Find the Pratt-Arrow risk-aversion functions for $0 < A < 100$ for the following utility functions, where A represents total assets in thousands of dollars:
(a) $U(A) = 1 - e^{-0.05A}$.
(b) $U(A) = \log A$.
Graph these risk-aversion functions and the risk-aversion function from Exercise 38 and compare them in terms of how the risk aversion changes as A increases.

(a) $U(A) = 1 - e^{-0.05A}$

$$\frac{dU}{dA} = 0.05e^{-0.05A} \qquad \frac{d^2U}{dA^2} = -0.0025e^{-0.05A}$$

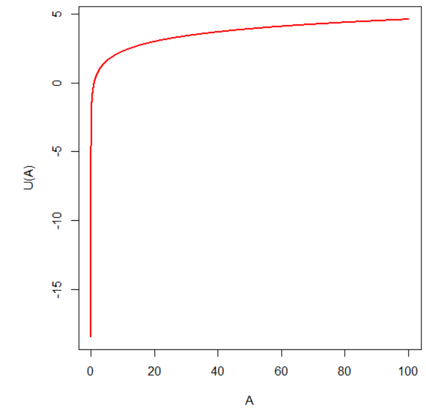


$$\Rightarrow r(A) = -\frac{-0.0025e^{-0.05A}}{-0.0025e^{-0.05A}} = 0.05$$

$\Rightarrow$ Risk aversion is constant (does not vary with the total assets)

(b) $U(A) = \log(A) = \ln(A)$

$$\frac{dU}{dA} = \frac{1}{A} \qquad \frac{d^2U}{dA^2} = -\frac{1}{A^2}$$



$$\Rightarrow r(A) = -\frac{-1/A^2}{1/A} = \frac{1}{A}$$

$\Rightarrow$ Risk aversion decreases with total assets

Multiple attributes

The consequences of the combination of each action a in a set of actions $\mathcal{A}$ and each state of the world $\theta \in \Theta$ may involve several attributes $\Rightarrow$ There are so-called *multiattribute utilities*.

Example

Assume you enjoy watching football, and you should decide between watching a football match at the television at home or live at the arena.

Watching at home comes with no additional cost, you will view the game from many perspectives, but you will miss the ‘atmosphere’.

Watching live at the arena comes with a cost (ticket, travel), your view is limited by the position of your seat, but you will feel the atmosphere.

Hence the utilities are in three attributes: {cost, view, atmosphere}

It is generally difficult to find a utility function that involves several attributes and still fulfils the axioms of utilities.

Additive utility model

Use the utilities for each attribute and sum them. In the example the utility function would then be

$$U(\text{cost}) + U(\text{view}) + U(\text{atmosphere})$$

Problem: Simply adding the utilities would give them equal weights. Is that wise?

Cash equivalents

The equal weights problem may be resolved if the utilities of the different attributes could be replaced by cash equivalents.

Would that be possible for U(cost)? U(view)? U(atmosphere)?

Cash equivalents *may* be found by (again) considering a choice between two options, for example:

> Option I: Obtain x units of money for certain
> Option II: Feel the atmosphere in the arena when a match is played

The value of x for which you are indifferent between the two options is the cash equivalent for Option II.

This is actually quite a recurrent consideration, but rather in weighing the experience of awful things against receiving money: "How much do I need to pay you for swimming in 2°C water?"

Weighted additive utility model

Instead of adding utilities of different attributes with equal weights a *weighted* sum can be used. For k different attributes the utility of taking action a with state of the world θ can be calculated as

$$U(a,\theta) = \sum_{i=1}^{k} w_i \cdot U_i(a,\theta)$$

where $w_1, \ldots, w_k$ are weights, *but* not necessarily summing to 1 (when should they?), and $U_i(a,\theta)$ is the utilitity of attribute i with action a and state of the world θ.

In the example, assume that you appreciate the atmosphere twice as much the multiple view perspectives, but just as much as the “no cost” and the “no cost” twice as much as the multiple view perspectives. Then a weighted additive utility model may be

$$2 \cdot U(\text{cost}) + U(\text{view}) + 2 \cdot U(\text{atmosphere})$$