Meeting 8:
Utility functions

Utility as a function of payoff

In some accounts for decision theory the utility function is assumed to be linear of payoff, i.e.

$$U(a,\boldsymbol{\theta}) = c + d \cdot R(a,\boldsymbol{\theta})$$

with $d > 0$.

Then, maximising the expected utility is equivalent to maximising the expected payoff.

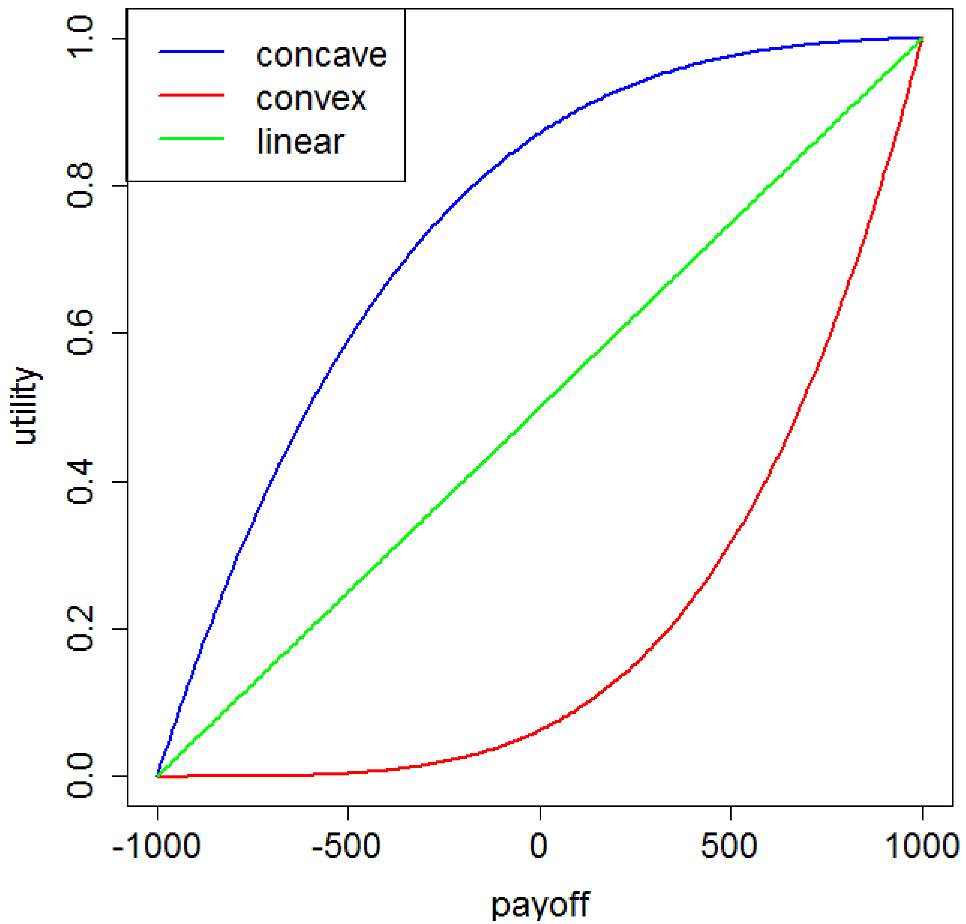
With such an assumption it is not possible to explain decisional behaviour among several individuals if they do not all adhere to the criterion of maximising the expected payoff.

Therefore, it is more general to assume that utility can be written as a function of payoff, but the function needs not to be linear:

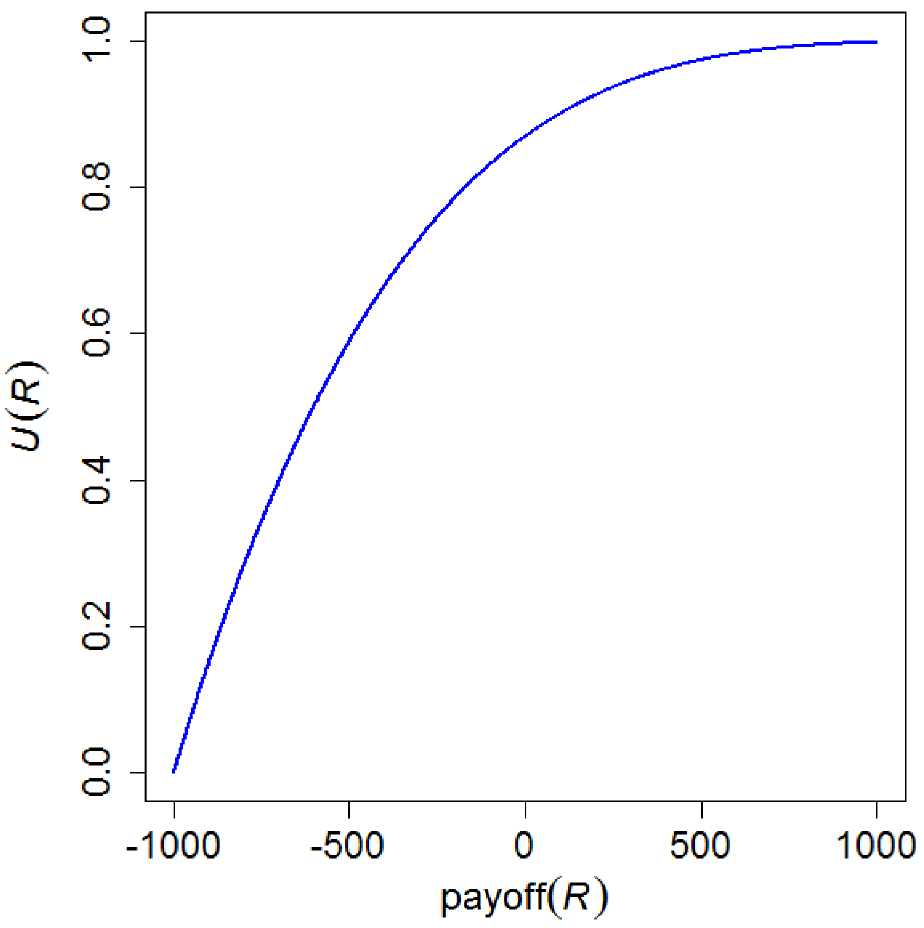
$$U(a,\boldsymbol{\theta}) = h\,(R(a,\boldsymbol{\theta}))$$

Besides the possibility that $U(a, \boldsymbol{\theta})$ can be linear in $R(a, \boldsymbol{\theta})$ the most common types of functions are

- $U(a, \boldsymbol{\theta})$ is a concave function of $R(a, \boldsymbol{\theta})$
- $U(a, \boldsymbol{\theta})$ is a convex function of $R(a, \boldsymbol{\theta})$



Concave utility functions



This functional form of the utility function characterizes a *risk avoider* (or *risk averse* decision maker).

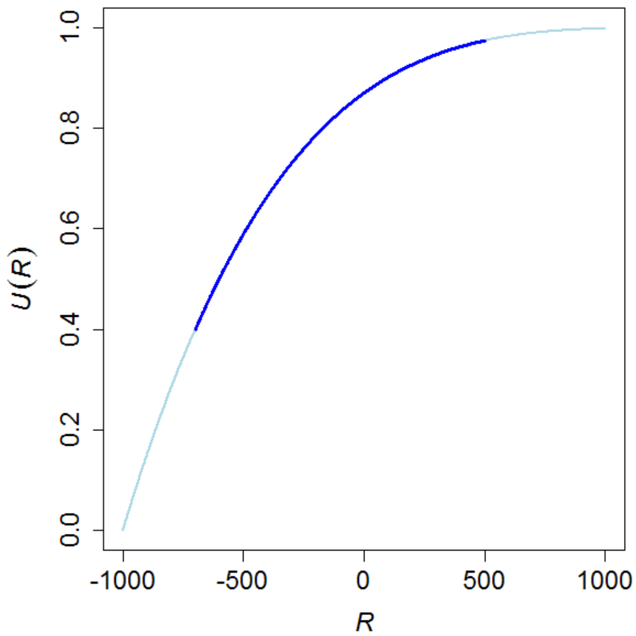
Why is it so?

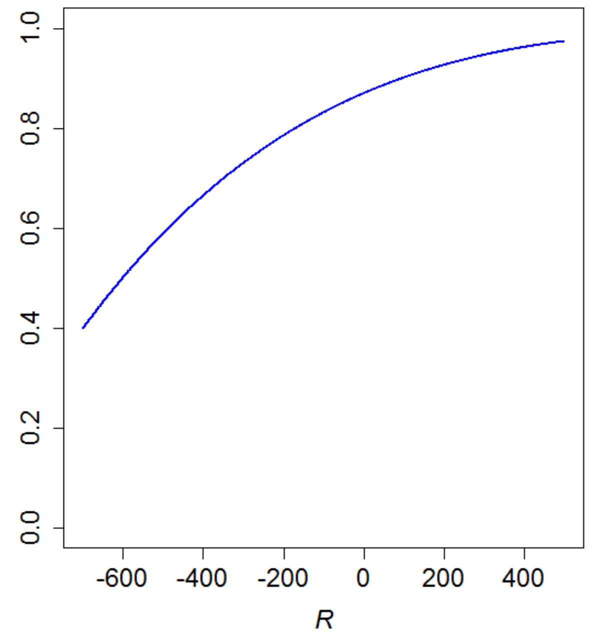
Consider the following bet:

Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

There are two actions: a_1 = "take the bet" and a_2 = "do not bet"

Focus on the range of money defined by the bet:





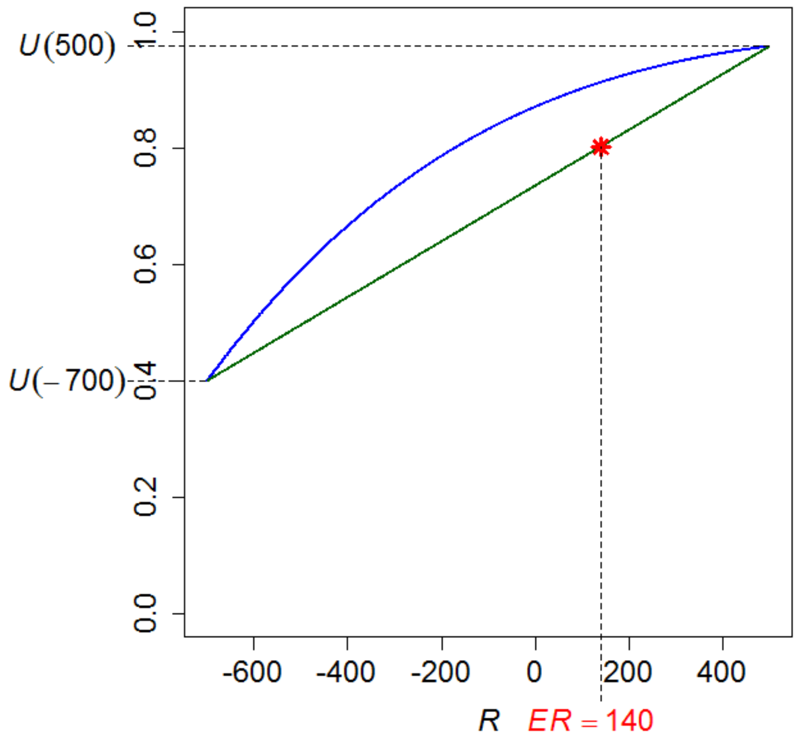
Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

Now, the expected payoff of the bet is

$$ER = 500 \cdot 0.7 - 700 \cdot 0.3 = 140$$

which is a convex combination of 500 and –700.

Since any convex combination of two points v_1 and v_2, i.e. any $v = p \cdot v_1 + (1-p) \cdot v_2$ where $0 < p < 1$, lies on the segment joining v_1 and v_2 (in one dimension this means $v_1 < v < v_2$) we can represent the expected payoff as a point on a straight line joining the points (–700, U(–700)) and (500, U(500)):



To clarify: the “two points” in this sense are points on the curve defining the utility function.

Hence, v_1 and v_2 both need to be on the curve, which then automatically defines the coordinates of the points.

The straight line corresponds with a utility function that is linear in payoff $\Rightarrow$ values along the line can be interpreted as payoff expressed in the same unit (scale) as $U(R)$.

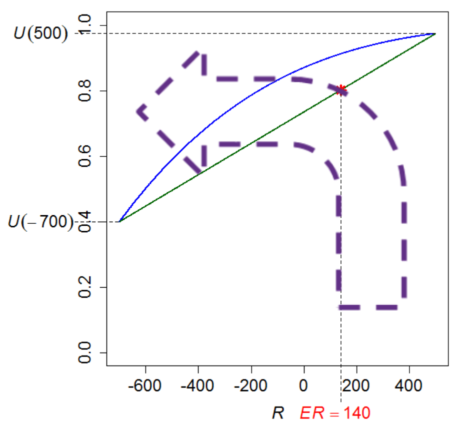
Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

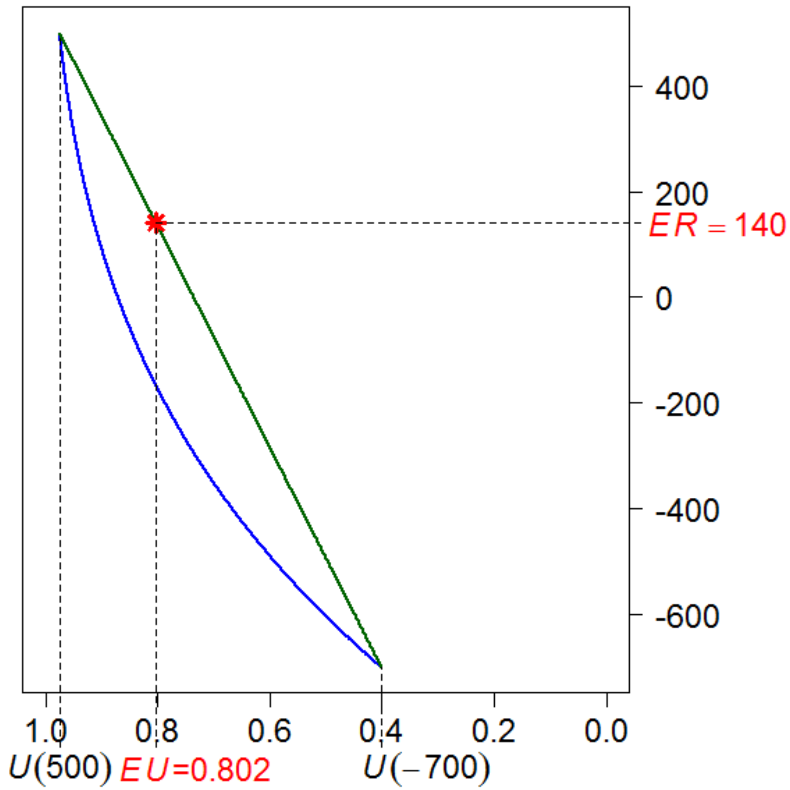
Now, it also holds that the expected utility (for money) of taking the bet is

$$EU(\text{bet}) = U(500)\cdot 0.7 + U(-700)\cdot 0.3$$

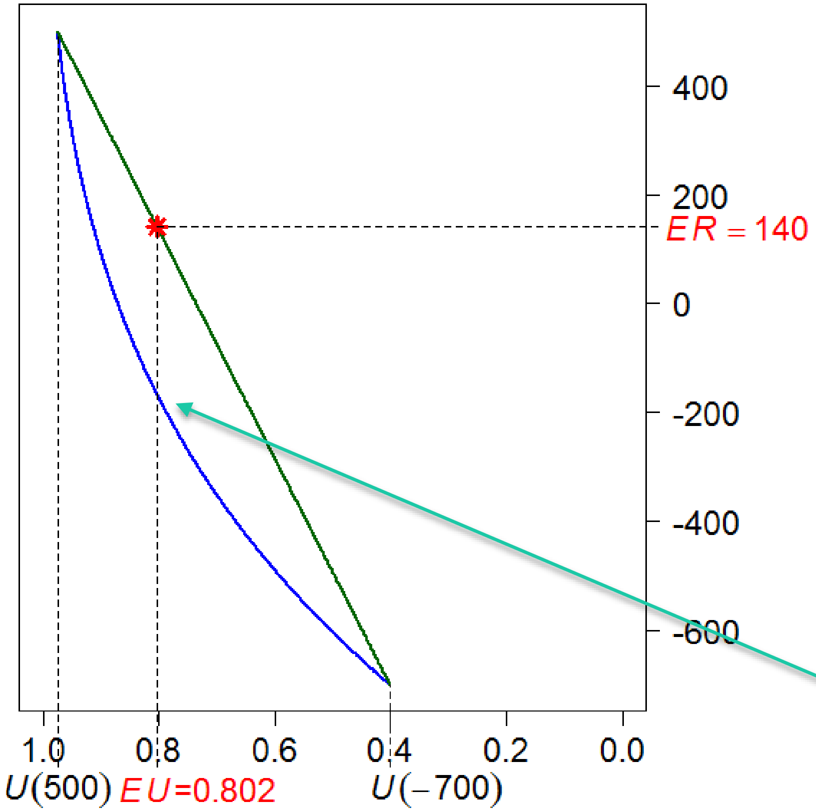
This is also a convex combination, but here of two utilities. This must be represented by the same point as was the corresponding convex combination of payoffs, but now we should view it from the "utility" perspective

Using the mathematical function that was used to produce the curve we can calculate $EU \approx 0.802$





Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3



Plotting R as a function of $U(R)$ we can see which cash equivalent (expressed on the same scale as $U(R)$) corresponds with which utility.

Here we can see that the cash equivalents for the utility of taking this bet are all lower than or equal to the payoff of taking the bet.

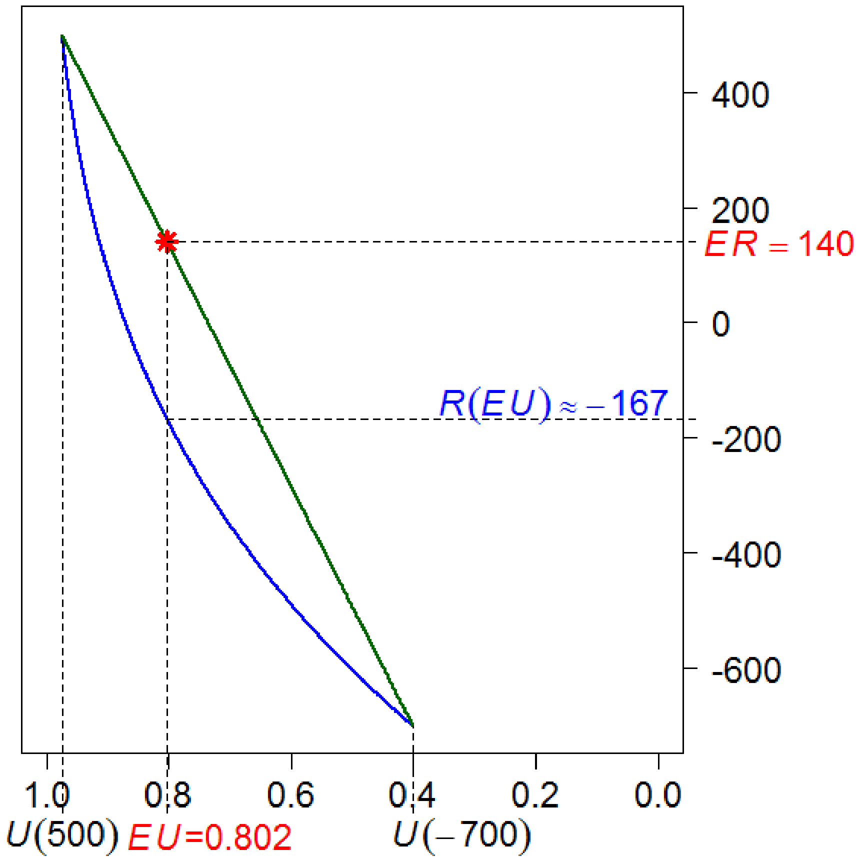
In particular, the value of EU is equal to the utility of a cash equivalent satisfying

$$R(EU) = U^{-1}(EU)$$

where U^{-1} is the inverse function of $U(R)$ restricted to $R \in (-700, 500)$

Here, again using the mathematical function "behind" the curve, we can calculate $R(EU) \approx -167$.

Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3



Hence, the decision maker appreciates the expected utility of taking the bet to be equivalent to a payoff of SEK –167 …

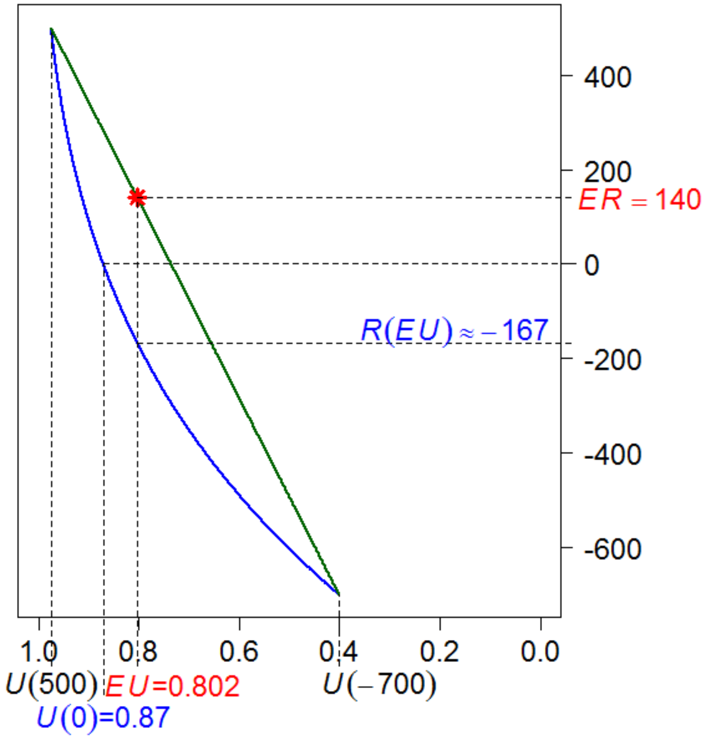
… while the expected payoff of taking the bet is SEK 140.

Now, the expected payoff of *not* taking the bet is (always) SEK 0.

The expected utility of not taking the bet must be equal to the utility when $R = 0$, i.e. $U(0)$.

This can again be calculated using the mathematical function behind $\Rightarrow U(0) \approx 0.87$

Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3



Now, the very decision problem is about taking the bet or not. The expected utility of taking the bet is 0.802 and the expected utility of not taking the bet is the utility corresponding with a payoff of SEK 0, which is 0.87.

Thus the optimal decision with the *EU*-criterion is to <u>not</u> take the bet.

R(*EU*) is called the *certainty equivalent* of the decision-maker. This is the lowest value of a certain payment that the decision-maker would prefer versus taking the bet. Apparently this value can be negative which would render a cost and not a return for the decision-maker.

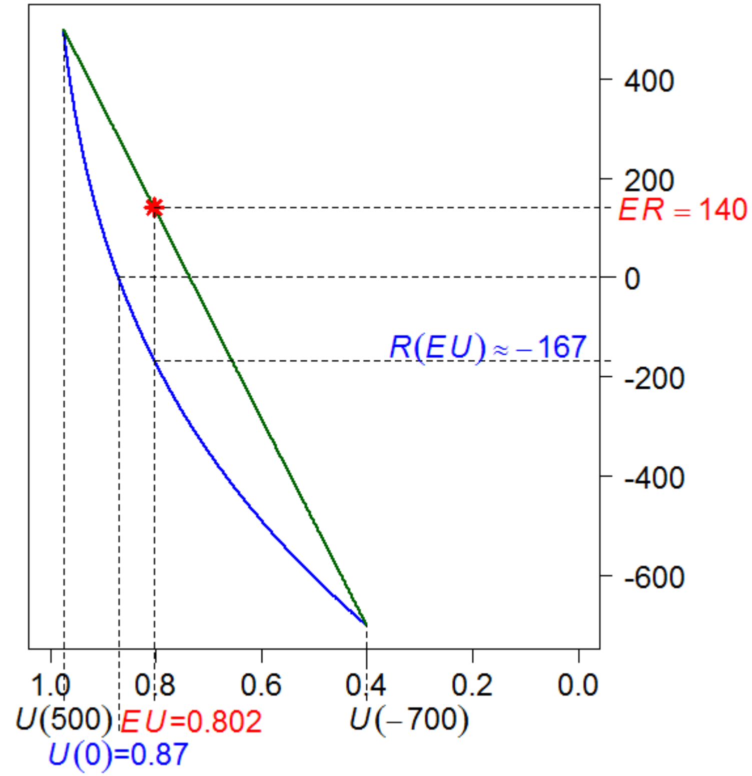
Alternatively expressed, the decision-maker is in this case indifferent between

1. Obtaining SEK –167 for certain *and*
2. Obtain SEK 500 with probability 0.7 and losing SEK 700 with probability 0.3

Risk premium

Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

When a risk avoider is exposed to a "bet", he or she will always have an expected utility of taking the bet that is lower than (or at highest equal to) the expected utility that is linear in payoff.



The payoff equivalent to the expected utility of this risk avoider is their certainty equivalent, *CE*, and the difference between the expected payoff, *ER*, and the certainty equivalent is called their *risk premium*, *RP*.

$$RP = ER - CE$$

In the above example the risk premium of the decision-maker is then approximately SEK 140 – (–167) = 307

Notice (again) that all functions and specific quantities are ***personal*** *to the decision-maker!*

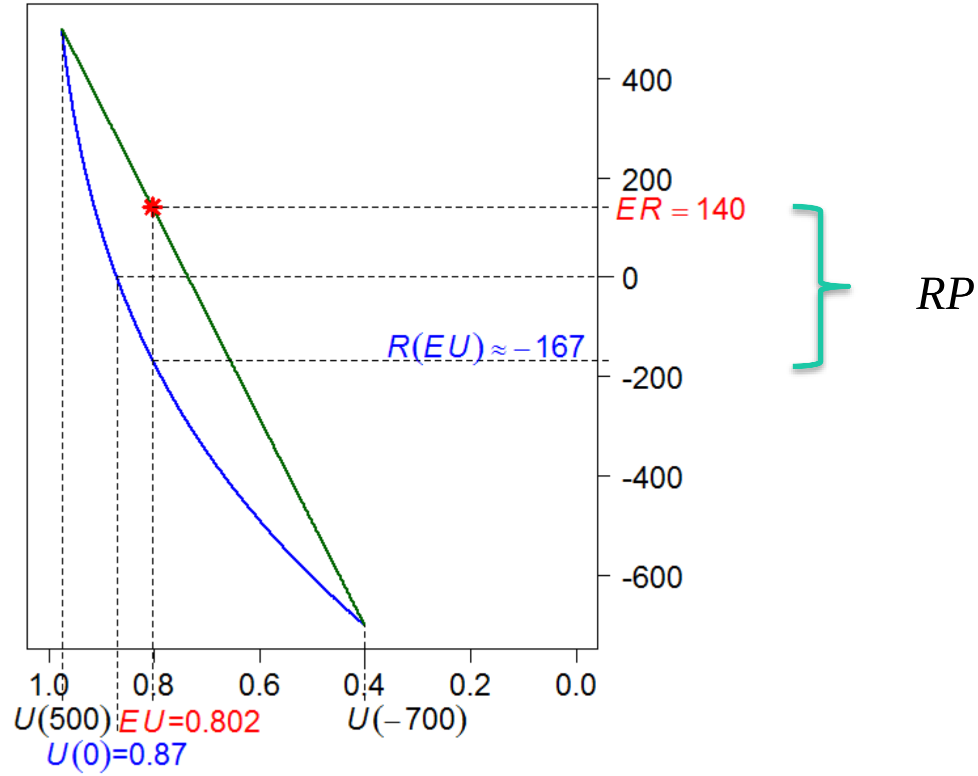
Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

What is then the difference between the certainty equivalent and the risk premium?

In the example above, the expected payoff was positive (*ER* = 140) while the certainty equivalent was negative (*CE* = –167).

The certainty equivalent is what the decision maker considers to be the expected utility in monetary terms of taking the bet. Hence they will never consider taking a bet with a negative certainty equivalent, but it would not generally suffice with a positive certain equivalent either.

The risk premium tells how much money must at least be *additionally paid* to the decision maker for making them take the bet. In the example above that amount was SEK 307.

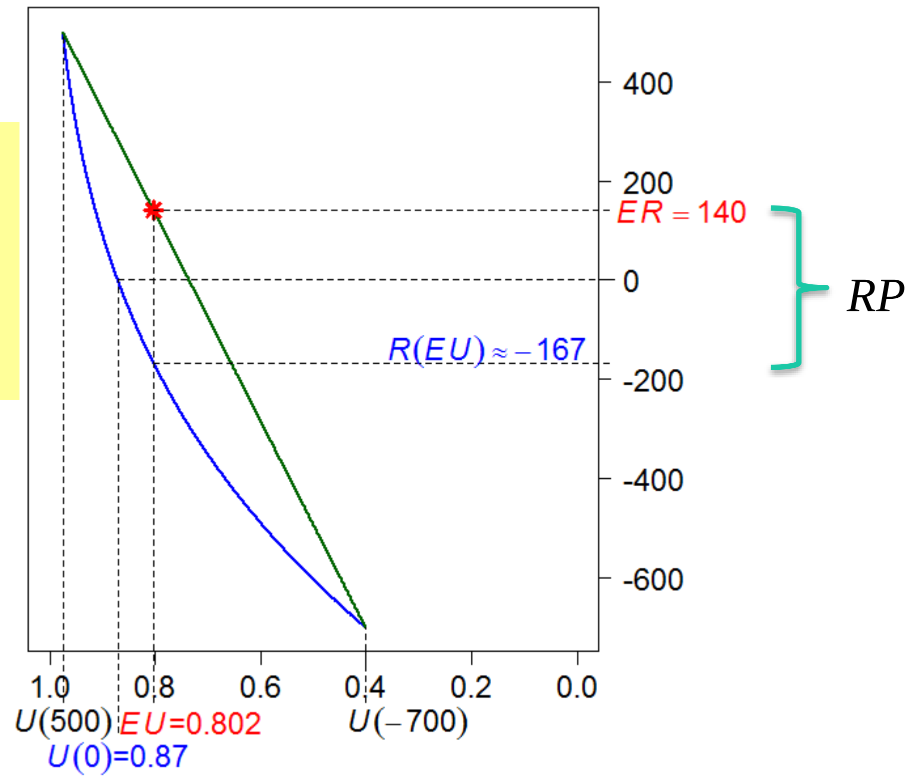


Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

It is the shape of the utility function that implies that the decision-maker does not become indifferent between

1. Obtaining SEK *x* for certain *and*
2. Obtain SEK 500 with probability 0.7 and losing SEK 700 with probability 0.3

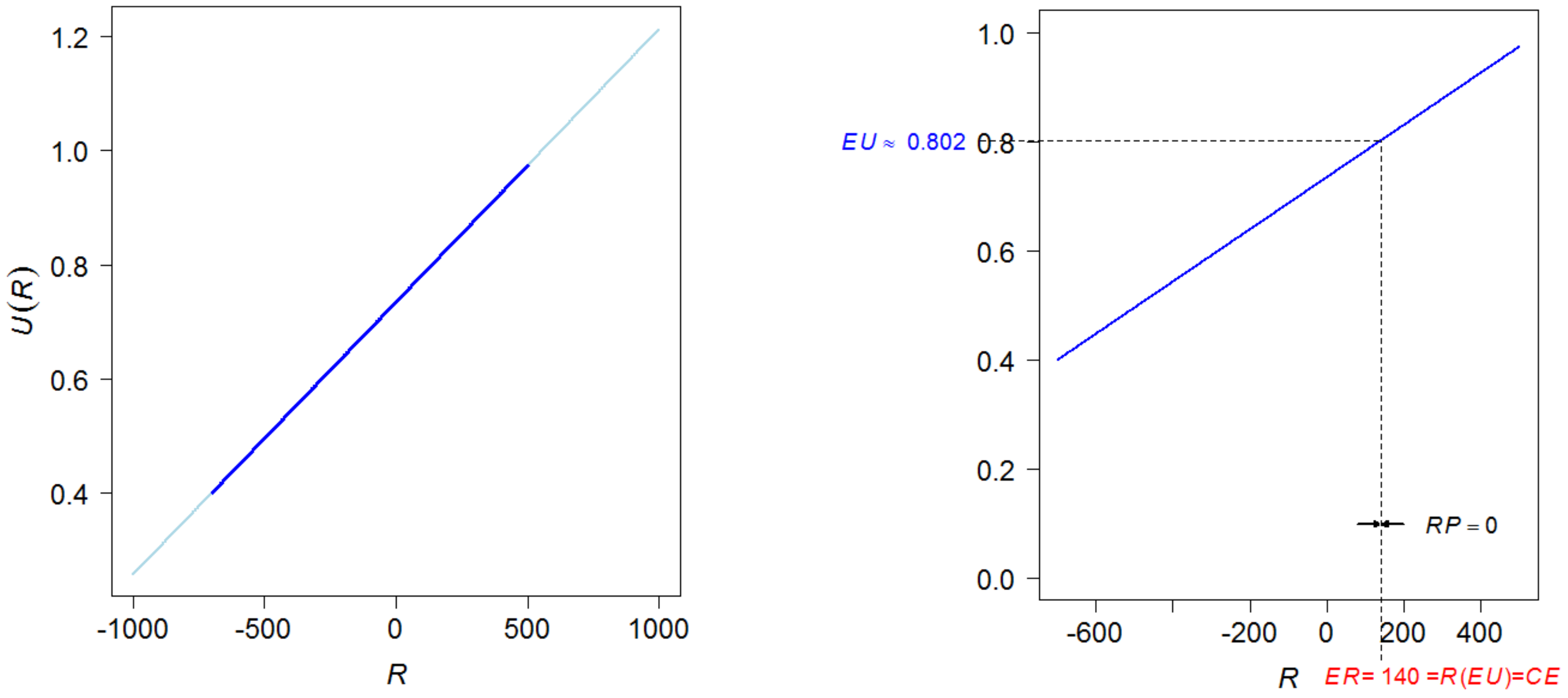
until *x* = *RP*.



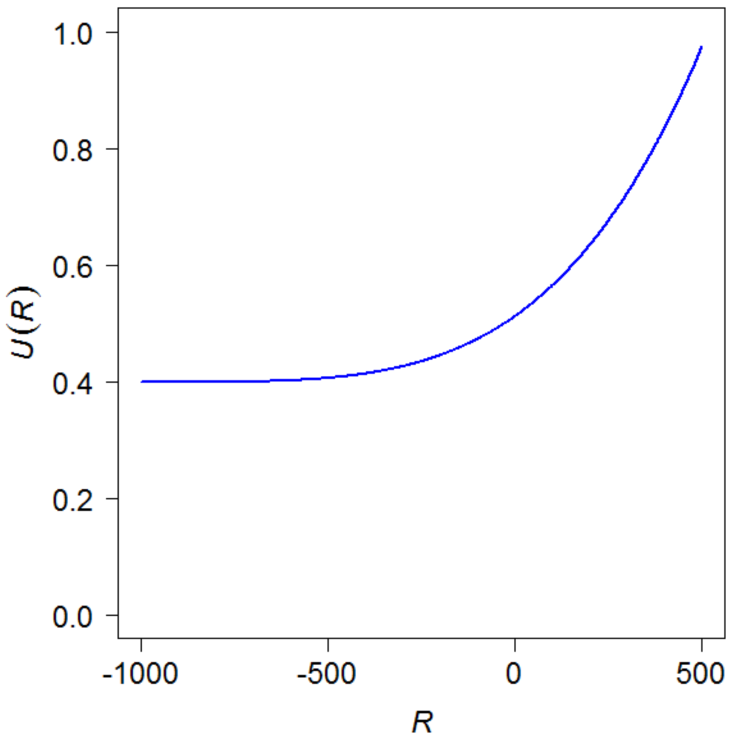
Hence, the higher the certainty equivalent the lower the risk premium.

If the expected payoff of a bet is 0, the bet is said to be a *fair* bet. In that case the risk premium will be equal to the certainty equivalent in absolute sense since
RP = 0 – *CE*.

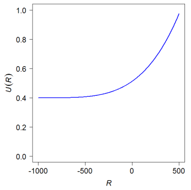
Now, if the decision-maker is *risk neutral* the expected utility for money of a bet coincides with the expected payoff (the utility is linear in payoff). This means that the certainty equivalent is always equal to *ER* and the risk premium is zero.



For a *risk taker* the situation is the opposite as for the risk avoider. The utility function of a risk taker is convex, e.g.



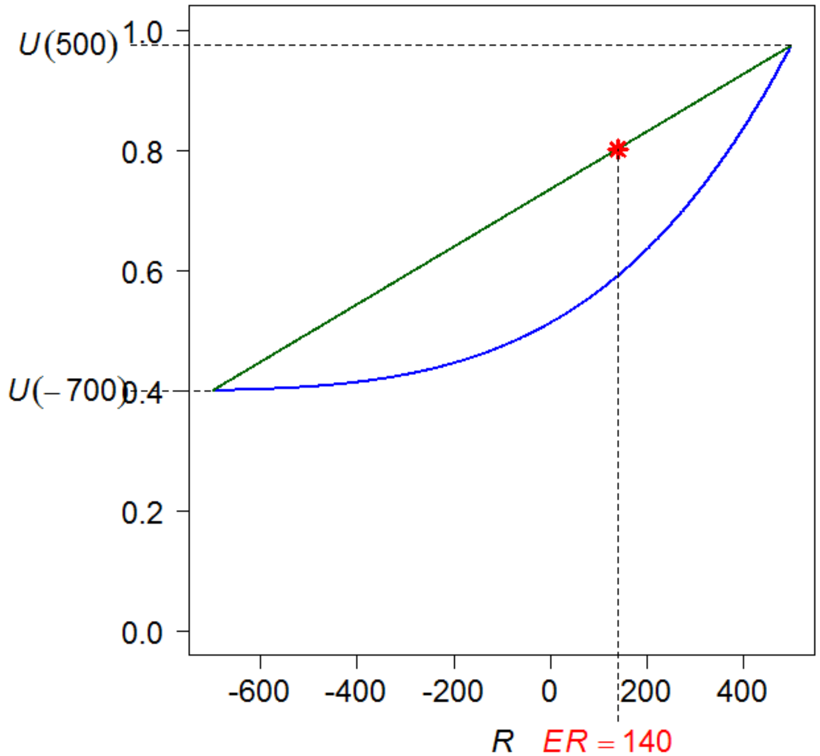
This function is here made such as the utilities for $R = -700$ and $R = 500$ are the same as with the previous utility function of a risk avoider.



Hence with the same bet as before, i.e.

Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

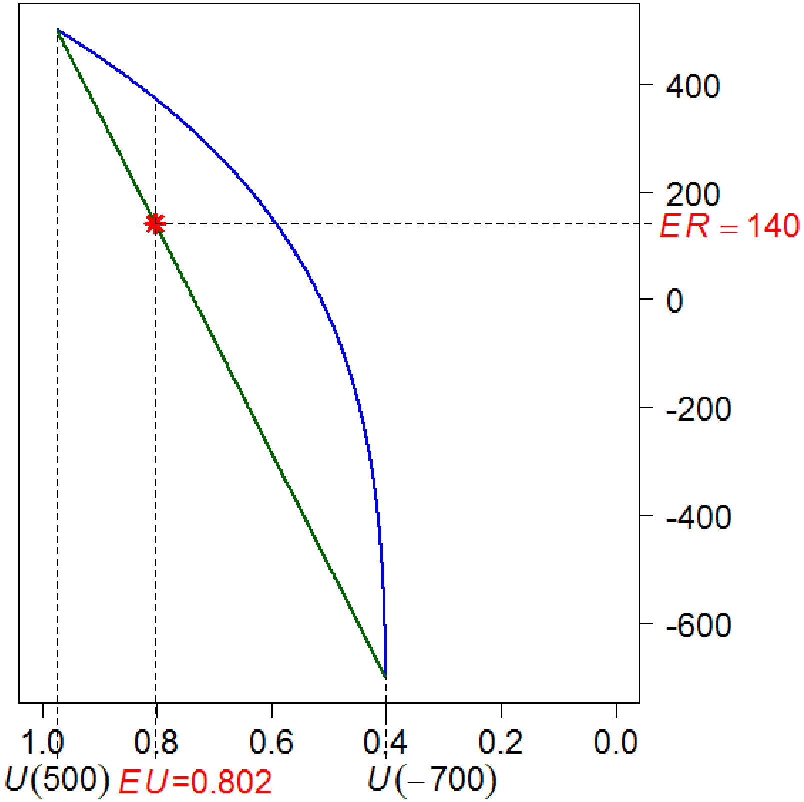
we can graphically illustrate this as



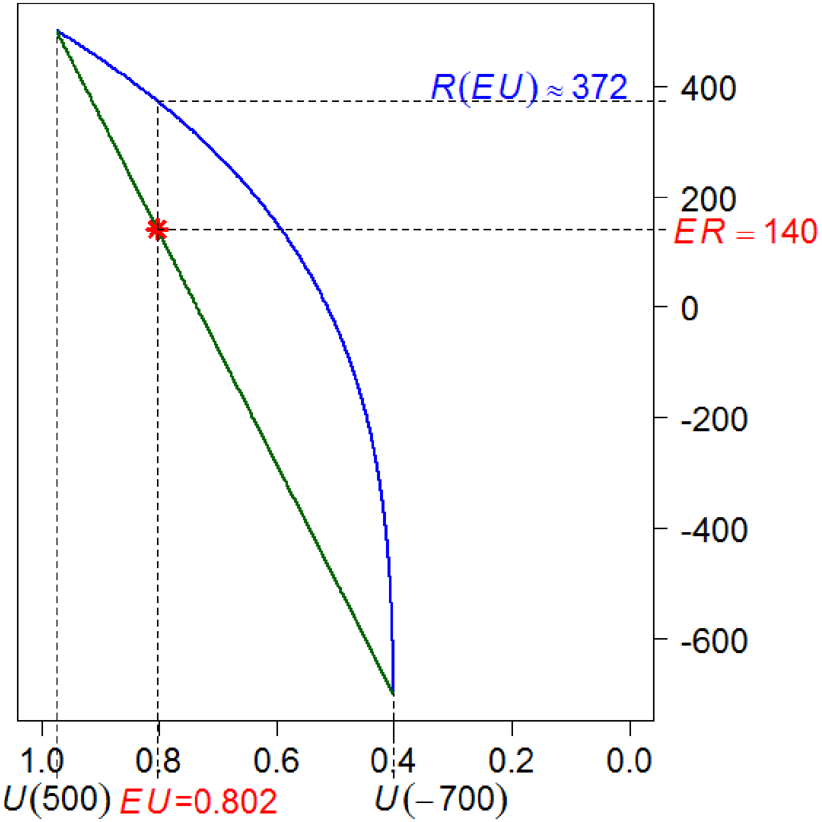
Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3

The value of *EU* for taking the bet is the same here as before, i.e. 0.802

If we – as previously – plot *R* against *U*(*R*) we obtain:

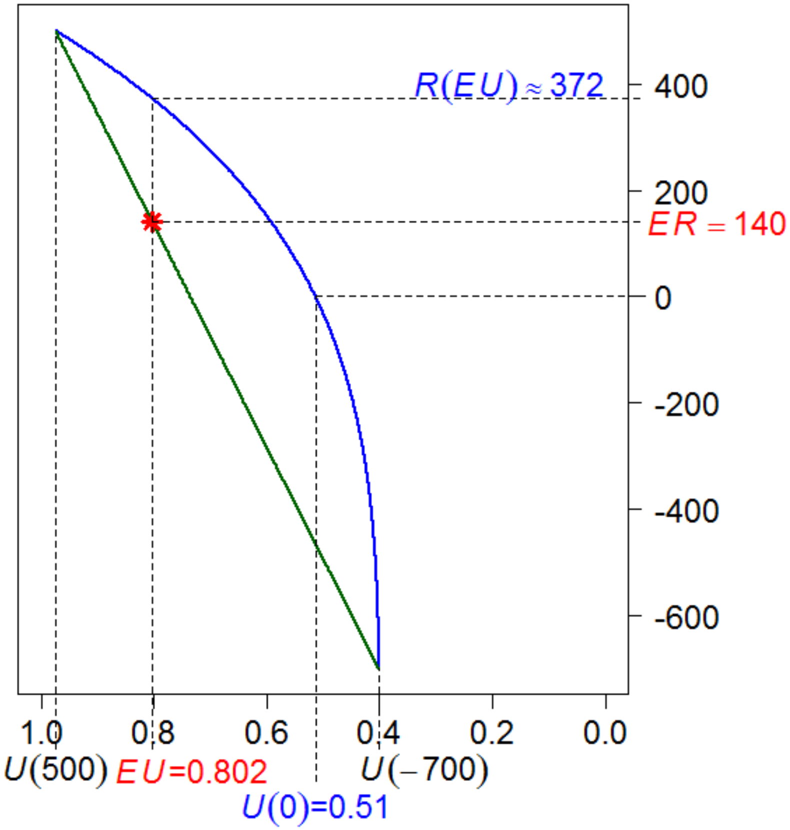


…and the certainty equivalent here becomes $CE = R(EU) \approx 372$



The expected utility of not taking the bet is calculated as $U(0) \approx 0.51$

Win SEK 500 with probability 0.7 and lose SEK 700 with probability 0.3



Thus the optimal decision with the *EU*-criterion is to take the bet, since $U(0) < EU$.

The risk premium with this utility function becomes

$$RP = ER - CE \approx 140 - 372 = -232 \text{ (SEK)}$$

…hence negative!

For a risk taker the certainty equivalent is always higher than (or at least equal to) the expected payoff.

Hence, the risk premium for taking a bet is always negative for a risk taker.

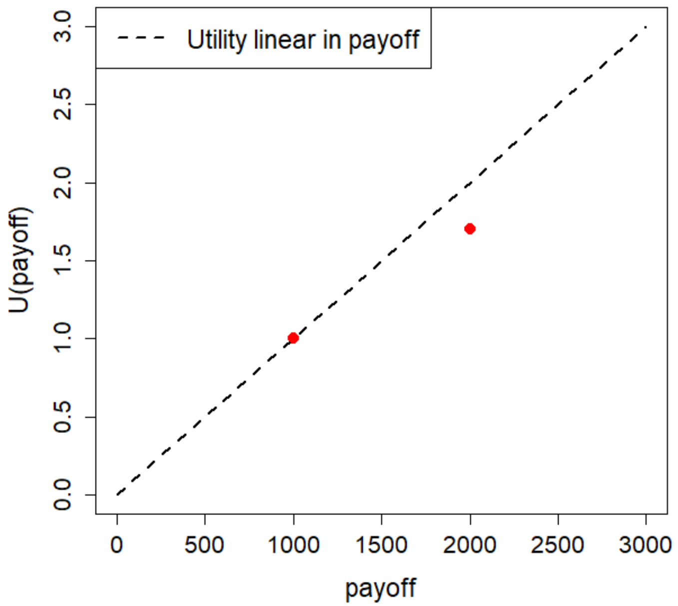
This means that a risk taker is willing to pay a certain amount for taking the bet.

Example

Assume you think that obtaining an amount of EUR 1000 is “good” but obtaining an amount of EUR 2000 is not twice as good as obtaining EUR 1000 but 70% “better”.

Would you be a risk avoider, risk neutral or risk taker in a bet?

For simplicity, let $U(\text{EUR } 1000) = 1 \Rightarrow U(\text{EUR } 2000) = 1.7$



Are the two red points on a concave or convex curve with respect to the straight line?

Example The TRISS scratch card

Ticket price: SEK 30

Payouts for a batch of 2 000 000 tickets

Payout (SEK)	Number of tickets
2 765 000	1
1 000 000	1
265 000	6
200 000	1
100 000	2
50 000	3
20 000	8
10 000	46
5 000	30
2 500	15
2 000	50
1 500	80
1 000	760

Payout (SEK)	Number of tickets
900	60
750	50
600	200
500	200
450	125
300	930
180	1 200
150	3 760
120	7 200
90	26 000
60	200 353
30	188 519

$\Rightarrow$ 429 600 tickets with winnings

Expected payout:

$$2765000 \cdot \frac{1}{2000000} + 1000000 \cdot \frac{1}{2000000} + 265000 \cdot \frac{3}{2000000} + \cdots$$

$$\cdots + 30 \cdot \frac{188519}{2000000} + 0 \cdot \frac{1570400}{2000000} = 15$$

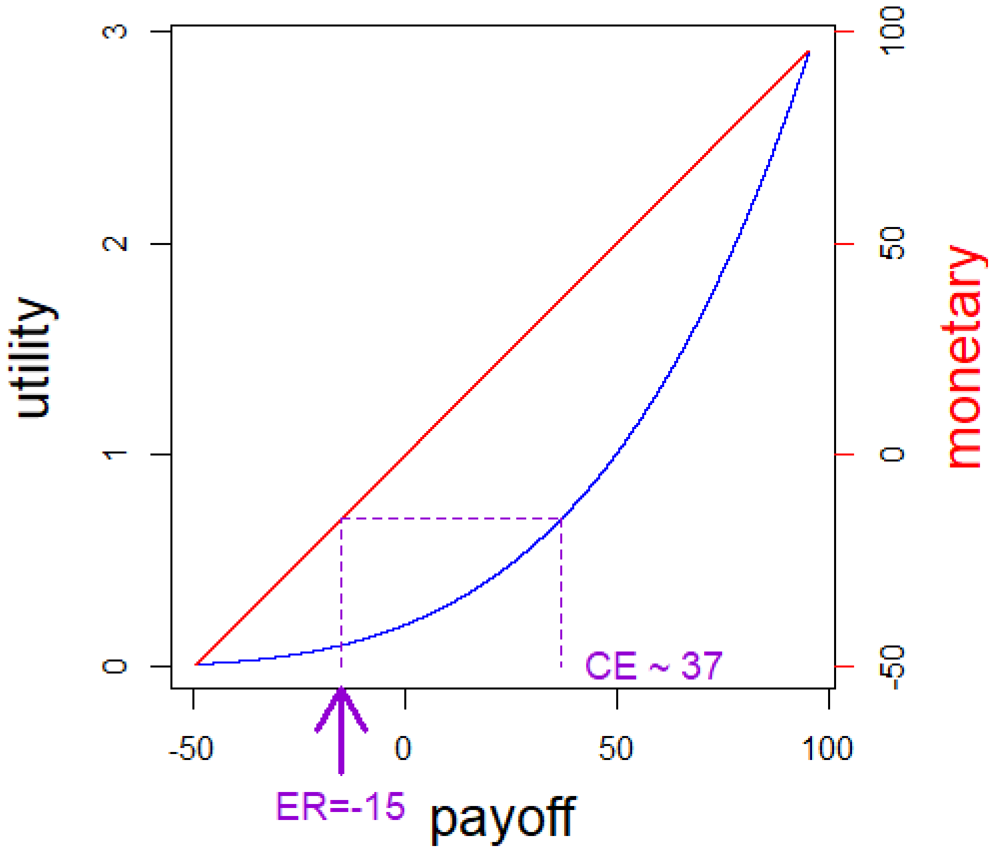
Expected payoff:

$$ER = 15 - 30 = -15$$

Would a risk avoider buy a ticket?

Would a risk neutral buy a ticket?

Would a risk taker buy a ticket?



utility
3
2
1
0
100
50
0
-50
monetary
CE ~ 37
-50
0
50
100
ER=-15
payoff