Meeting 17:
Game theory

Decision theory – Game theory Difference?

- The theory so far presented applies to decision making by <u>one</u> decision maker. The states of the world follow a probability distribution elicited by that decision maker.

- Games comprise at least two decision makers (players). The states of the world depends on how each decision maker plays.

- Actions to be taken by one player of a game must be according to a *strategy* based on how the other players would play.

- Note that a single player needs not to be a single person, it can be a group of persons, a company etc.

Taxonomy of games

- **Zero-sum games**/Nonzero-sum games
  - In a zero-sum game, what one players wins is what the other player(s) lose. The total amount of utility is fixed.
- **Non-cooperative games**/Cooperative games
  - In a cooperative game there are legal rules or similar that prevents players from deviating from a strategy that has been agreed on. If no such rules exist, the game is non-cooperative (but the players may still cooperate).
- **Simultaneous-move games/**Sequential-move games
  - In a simultaneous-move game no player knows the other player’s strategy in advance – has however not do with the time-point the player choose their action.
  - In sequential-move games actions are taken in subsequent rounds, and one player has partial or full information about the strategy chosen by the other player(s) in the previous round.
  - Games with *perfect information* are sequential-move games where each player has <u>full</u> information about the other player’s strategy in the previous round (Example: Chess – strategy here means move).

- **Symmetric games**/non-symmetric games
  - A symmetric games is a game in which all players face the same strategies (actions) and outcomes (Example: *Prisoner's dilemma*).
- **Two-person games**/*n*-person games
  - Two-person games are played by two opponents (but each opponent can be a group of several people). These games are much easier to analyse than *n*-person games. The latter can for example not be easily illustrated graphically (like with a decision matrix).
- **Pure strategy games**/Mixed strategy games
  - In a pure strategy game, a player chooses the strategy that is optimal (for them). In a mixed strategy game, the player with choose the optimal strategy with a probability p. Using a random device will then be need to make the choice. If there are only two strategies, the non-optimal will be chosen with probability $1 – p$. With several alternatives, there must be some ordering and assignment of probabilities to these.
- **Non-iterated game**/Iterated game
  - A non-iterated game is played once. An iterated game is played several times and the players can successively learn and change strategies.

Solutions to games – Dominance principle

The *dominance principle* means that in a choice between two strategies (possibly out of more than two), if one strategy dominates the other the former will be chosen.

To apply the principle, three assumptions must hold:

1. All players are rational – try to play strategies that best promote the objective that is important to them.
2. All players <u>know</u> that the other players are rational –*common knowledge of rationality* (CKR) .
3. The dominance principle is a valid principle of rationality

However, for many games, the dominance principle cannot be used (because no dominant strategies exist).

Example

Assume there are two players, A and B of a game. Each player chooses between 3 strategies. For player A, these are denoted A1, A2 and A3, and for player 2 they are denoted B1, B2, and B3.

A *game matrix* shows the outcomes (payoff, utility) for each player for each pair of strategies.

		Player B strategies		
		B1	B2	B3
Player A, strategies	A1	(A_{11}, B_{11})	(A_{12}, B_{12})	(A_{13}, B_{13})
	A2	(A_{21}, B_{21})	(A_{22}, B_{22})	(A_{23}, B_{23})
	A3	(A_{31}, B_{31})	(A_{32}, B_{32})	(A_{33}, B_{33})

A_{ij} is the outcome for player A and B_{ij} is the outcome for player B when strategies Ai and Bj are played.

Assume the following game matrix

		Player B strategies		
		B1	B2	B3
Player A, strategies	A1	$(0, 1)$	$(2, 2)$	$(1, 0)$
	A2	$(0, 0)$	$(5, 3)$	$(1, 4)$
	A3	$(1, 0)$	$(3, 3)$	$(2, 6)$

First, consider Player A's choices.

Whatever strategy Player B will play, A1 is dominated by both A2 and A3
$\Rightarrow$ Both players know that Player A will not play A1, but A2 or A3

Thus, Player B should compare their strategies given Player A plays A2 or A3.

Strategy B3 will then dominate both B1 and B2, so the obvious strategy for Player B is B3.

With that conclusion, Player A should play A3, since A3 dominates A2 given that Player B plays B3.

$\Rightarrow$ The strategy pair (A3,B3) is the solution to the game using the dominance principle

Two-person zero-sum games

Example Assume the following game matrix for the two players A and B in the previous example:

	B1	B2	B3
A1	$(4, -4)$	$(-3, 3)$	$(-7, 7)$
A2	$(5, -5)$	$(3, -3)$	$(7, -7)$
A3	$(3, -3)$	$(2, -2)$	$(-1, 1)$

For every pair of strategies, the outcome for Player B is the negative of the outcome for Player A – this is what characterizes a zero-sum game.

Hence, it is sufficient to provide the outcomes for one of the players in the matrix:

Using Player A's outcomes:

	B1	B2	B3
A1	4	-3	-7
A2	5	3	7
A3	3	2	-1

	B1	B2	B3
A1	$(4, -4)$	$(-3, 3)$	$(-7, 7)$
A2	$(5, -5)$	$(3, -3)$	$(7, -7)$
A3	$(3, -3)$	$(2, -2)$	$(-1, 1)$

Assume Player B considers that Player A will play A1. Then the best strategy for Player B would be B3.

However, Player A would then know that, which means she/he will get bad out of using that strategy. Hence Player A would not Play A1.

Now, assume Player B considers that Player A will play A2. Then Player B would play B2, since that gives them the best outcome (although negative).

Player A again knows that. However, if Player B plays B2, then the best strategy for Player A is A2.

Finally, assume Player B considers that Player A will play A3. Then the best strategy for Player B is B3. But with Player B playing B3, the best strategy for Player A is A2.

The pair of strategies (A2,B2) is therefore said to be in *equilibrium*.

Definition of equilibrium:

A pair of strategies are in equilibrium if and only if it holds that once this pair is chosen, none of the players could reach a better outcome by unilaterally switching to another strategy.

Note! There can be several pairs in equilibrium.

Finding a pair in equilibrium

The minimax condition for a two-person zero-sum game:

A pair of pure strategies are in equilibrium if (but not only if) the outcomes determined by the strategies equal
(minimum of the outcomes with Person 1's strategy,
maximum of the negatives of the outcomes with Person 2's strategy)

Example cont.

	B1	**B2**	**B3**
A1	$(4, -4)$	$(-3, 3)$	$(-7, 7)$
A2	$(5, -5)$	$(3, -3)$	$(7, -7)$
A3	$(3, -3)$	$(2, -2)$	$(-1, 1)$

Using mixed strategies

Example Consider the following game matrix

	B1	B2	B3
A1	$(-20, 20)$	$(20, -20)$	$(0, 0)$
A2	$(20, -20)$	$(0, 0)$	$(-20, 20)$
A3	$(0,0)$	$(-20, 20)$	$(20, -20)$

Using the minimax condition, there is no equilibrium of pure strategies.

Now, consider the mixed strategies:

Player A:

Strategy	Probability
A1	p_1
A2	p_2
A3	p_3

Player B:

Strategy	Probability
B1	q_1
B2	q_2
B3	q_3

The expected outcomes for Player A for each of Player B's strategies are

B1: $(-20) \cdot p_1 + 20 \cdot p_2 + 0 \cdot p_3 = E(A|\text{B}1)$

B2: $20 \cdot p_1 + 0 \cdot p_2 + (-20) \cdot p_3 = E(A|\text{B}2)$

B3: $0 \cdot p_1 + (-20) \cdot p_2 + 20 \cdot p_3 = E(A|\text{B}3)$

If $E(A|\text{B}1) = E(A|\text{B}2) = E(A|\text{B}3)$ Player A' expected outcome will not depend on Player B's strategy.

For this to happen, it is required that

$$20(p_1 - p_2) = 20(p_1 - p_3) = 20(p_3 - p_2)$$

with solution $p_1 = p_2 = p_3 = 1/3$

Since the game is symmetric, for Player B's expected outcomes to be equal independently od Player A's strategy it is required that

$q_1 = q_2 = q_3 = 1/3$

Hence, if Player A's (mixed) strategy is [A1 $1/3$, A2 $1/3$, A3 $1/3$], then Player B can *do no better* than play [B1 $1/3$, B2 $1/3$, B3 $1/3$].

Therefore, these two mixed strategies are in equilibrium.

The minimax theorem:

> Every two-person zero-sum game has a solution, i.e. there is always a pair of strategies that are in equilibrium, and if there is more than one pair, they all have the same expected utility.

Nonzero-sum games

Example revisited Prisoner's dilemma

Recall from Meeting 9:

Assume two perpetrators of two crimes (one serious, one less serious) are arrested.

They are put in different cells and cannot communicate with each other

The prosecutor gives each of the perpetrators the following information:

- "If you both deny to confess, you will each get two years in prison for the less serious crime."
- "If one of you denies and the other confesses, the former will get 20 years in prison, and the latter will get 1 year in prison (*thanks for confessing*)."
- "If you both confess, you will both get 10 years in prison."

Game matrix

		Prisoner 2	
		Confess	Deny
Prisoner 1	Confess	$(-10, -10)$	$(-1, -20)$
	Deny	$(-20, -1)$	$(-2, -2)$

This is a two-person, non-cooperative game, but not a zero-sum game. Hence, the minimax theorem cannot be applied.

Is there an equilibrium?

For Prisoner 1, confessing dominates denying, and the same holds for Prisoner 2. Hence (confess,confess) will be in equilibrium, because assuming the other prisoner will confess, any of the prisoners cannot do better than also confess.

This does not apply to any other pair of strategies.

The Nash equilibrium

John Nash defined an equilibrium concept in 1950 that can be worded different ways.

One alternative:

> “A Nash Equilibrium is a set of strategies that players act out, with the property that no player benefits from changing their strategy. Intuitively, this means that if any given player were told the strategies of all their opponents, they still would choose to retain their original strategy.” (*Katz*, *Williams*, *Strandberg*, https://brilliant.org/wiki/nash-equilibrium/)

Whatever formulation, the Nash equilibrium is equivalent to the definition given above.

Example Stag hunting (classical example originally discussed by Rousseau (the French philosopher)

Assume two hunters can either choose to cooperate in hunting stag or choose to individually hunt hare.

If they cooperate, a stag will be caught and sharing the meat they get about 12 kg of meat each.

If a hunter chooses to individually hunt hare, she/he will get about 2,5 kg of meat

Hence, if they first agree on hunting stag, but one of them changes his strategy hunting hare on themselves instead, he/she will get about 2,5 kg of meat, while the other gets nothing.

This is a two-person, non-cooperative, nonzero-sum game.

Game matrix

		Hunter 2	
		Stag	Hare
Hunter 1	Stag	(12, 12)	(0, 2,5)
	Hare	(2,5, 0)	(2,5, 2,5)

Is there any Nash equilibrium?

Neither of the hunters have a dominant strategy.

If the hunters cooperate in hunting stag and each hunter assumes the other hunter sticks to this agreement, none of them comes out better by breaking the agreement.

$\Rightarrow$ (Stag,Stag) is in Nash equilibrium

However, if they do not cooperate and one hunter assumes the other hunter will hunt hare, then the other hunter can do no better than hunting hare as well.

$\Rightarrow$ (Hare,Hare) is also in Nash equilibrium