Meeting 12: Value and cost of information

Two-stage decision problem

1. Decide whether new data should be acquired.
2. Choose an action based on the information available after 1.

The decision in stage 1 will be based on the *value* of the information that *may* be obtained from acquiring new data <u>and</u> the cost of obtaining these data.

Hence, the value of the information and the cost of acquiring it can be the components to *dimension* the sample size.

Decisive approach to sampling

Net gain of sampling

Since taking samples (obtaining a sample result) comes with a cost, the value of sample information is of interest when it is compared against the cost of sampling.

Note! This comparison requires that the utility and cost can be expressed in the same unit

Net gain of sampling given sample result y:

$$\text{NGS}(y) = \text{VSI}(y) - \text{CS}$$

where CS is the cost of sampling (not dependent on the sample result)

Expected net gain of sampling, ENGS:

$$\text{ENGS} = \text{EVSI} - \text{CS}$$

…as function of the sample size n :

$$\text{ENGS}(n) = \text{EVSI}(n) - \text{CS}(n)$$

$\Rightarrow$ Maximum sample size n_{max} must fulfil $\text{ENGS}(n_{\text{max}}) \geq 0$ and $\text{CS}(n_{\text{max}})$ within budget.

Exercise 6.20

20. In the automobile-salesman example discussed in Section 3.4, suppose that the owner of the dealership must decide whether or not to hire a new salesman. The payoff table (in terms of dollars) is as follows.

		STATE OF THE WORLD		
		Great salesman	*Good salesman*	*Poor salesman*
ACTION	*Hire*	60,000	15,000	-30,000
	Do not hire	0	0	0

The prior probabilities for the three states of the world are $P(\text{great}) = 0.10$, $P(\text{good}) = 0.50$, and $P(\text{poor}) = 0.40$. The process of selling cars is assumed to behave according to a Poisson process with $\tilde{\lambda} = 1/2$ per day for a great salesman, $\tilde{\lambda} = 1/4$ per day for a good salesman, and $\tilde{\lambda} = 1/8$ per day for a poor salesman.

(a) Find VPI(great salesman), VPI(good salesman), and VPI(poor salesman).

(b) Find the expected value of perfect information.

(c) Suppose that the owner of the dealership can purchase sample information at the rate of \$10 per day by hiring the salesman on a temporary basis. He must sample in units of four days, however, for a salesman's work week consists of four days. He considers hiring the salesman for one week (four days). Find EVSI and ENGS for this proposed sample.

	GREAT	GOOD	POOR
Hire	60,000	15,000	–30,000
Do not hire	0	0	0

(a) Since we are given a payoff table we compute VPI using the formula

$$VPI(\theta) = R(a_\theta, \theta) - R(a', \theta)$$

where a_θ is the optimal action when θ is the state of the world and a' is the (prior) optimal action with respect to maximised expected payoff.

$\tilde{\theta}$ can here assume the states "GREAT", "GOOD" and "POOR".

The prior probabilities for the three states of the world are $P(\text{great}) = 0.10$, $P(\text{good}) = 0.50$, and $P(\text{poor}) = 0.40$. The process of selling cars is assumed to behave according to a Poisson process

<u>Prior expected payoffs and prior optimal action:</u>

$$E(R(a = \text{"Hire"})) = 60000 \cdot P(\tilde{\theta} = \text{GREAT}) + 15000 \cdot P(\tilde{\theta} = \text{GOOD})$$

$$-30000 \cdot P(\tilde{\theta} = \text{POOR}) =$$

$$= 60000 \cdot 0.10 + 15000 \cdot 0.50 - 30000 \cdot 0.40 = 1500$$

$$E(R(a = \text{"Do not hire"})) = 0 \cdot 0.10 + 0 \cdot 0.50 + 0 \cdot 0.40 = 0$$

$\Rightarrow a'$ is "Hire".

	GREAT	GOOD	POOR
Hire	60,000	15,000	–30,000
Do not hire	0	0	0

If $\tilde{\theta}$ = GREAT or $\tilde{\theta}$ = GOOD action "Hire" is optimal,
if $\tilde{\theta}$ = POOR the action "Do not hire" is optimal.

$$\mathrm{VPI(GREAT)} = 60000 - 60000 = 0$$

$$\mathrm{VPI(GREAT)} = 15000 - 15000 = 0$$

$$\mathrm{VPI(POOR)} = 0 - (-30000) = 30000$$

(b) The expected value of perfect information is

$$\mathrm{EVPI} = \mathrm{VPI(GREAT)} \cdot P\left(\tilde{\theta} = \mathrm{GREAT}\right) + \mathrm{VPI(GOOD)} \cdot P\left(\tilde{\theta} = \mathrm{GOOD}\right)$$
$$+\mathrm{VPI(POOR)} \cdot P\left(\tilde{\theta} = \mathrm{POOR}\right) = 0 \cdot 0.10 + 0 \cdot 0.50 + 30000 \cdot 0.40 = 12000$$

	GREAT	GOOD	POOR
Hire	60,000	15,000	–30,000
Do not hire	0	0	0

(c) Let $\tilde{y}$ = Number of automobiles sold during one week.

The process of selling cars is assumed to behave according to a Poisson process with $\tilde{\lambda} = 1/2$ per day for a great salesman, $\tilde{\lambda} = 1/4$ per day for a good salesman, and $\tilde{\lambda} = 1/8$ per day for a poor salesman.

Then…

$$P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{GREAT}\right) = \frac{\left(4 \cdot (1/2)\right)^y}{y!} \cdot e^{-4\cdot(1/2)} = \frac{2^y}{y!} \cdot e^{-2}$$

$$P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{GOOD}\right) = \frac{\left(4 \cdot (1/4)\right)^y}{y!} \cdot e^{-4\cdot(1/4)} = \frac{1^y}{y!} \cdot e^{-1}$$

$$P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{POOR}\right) = \frac{\left(4 \cdot (1/8)\right)^y}{y!} \cdot e^{-4\cdot(1/8)} = \frac{0.5^y}{y!} \cdot e^{-0.5}$$

Now,

$$\text{EVSI} = \sum_{y} \text{VSI}(y) \cdot P(y)$$

$$\text{ENGS} = \text{EVSI} - \text{CS}$$

where $\quad \text{VSI}(y) = E(R(a''|y)|y) - E(R(a'|y)|y)$

In subtask (a) we obtained a' =“Hire”

The expected posterior payoffs are

$$\begin{aligned} E(R(\text{"Hire"}|y)|y) = {} & R\left(\text{"Hire"}, \tilde{\theta} = \text{GREAT}\right) \cdot P\left(\tilde{\theta} = \text{GREAT}\middle|\tilde{y} = y\right) \\ & + R\left(\text{"Hire"}, \tilde{\theta} = \text{GOOD}\right) \cdot P\left(\tilde{\theta} = \text{GOOD}\middle|\tilde{y} = y\right) \\ & + R\left(\text{"Hire"}, \tilde{\theta} = \text{POOR}\right) \cdot P\left(\tilde{\theta} = \text{POOR}\middle|\tilde{y} = y\right) \\ = {} & 60000 \cdot P\left(\tilde{\theta} = \text{GREAT}\middle|\tilde{y} = y\right) + 15000 \cdot P\left(\tilde{\theta} = \text{GOOD}\middle|\tilde{y} = y\right) \\ & - 30000 \cdot P\left(\tilde{\theta} = \text{POOR}\middle|\tilde{y} = y\right) \end{aligned}$$

$$\begin{aligned} E(R(\text{"Do not hire"}|y)|y) = \cdots = {} & 0 \cdot P\left(\tilde{\theta} = \text{GREAT}\middle|\tilde{y} = y\right) \\ & + 0 \cdot P\left(\tilde{\theta} = \text{GOOD}\middle|\tilde{y} = y\right) - 0 \cdot P\left(\tilde{\theta} = \text{POOR}\middle|\tilde{y} = y\right) = 0 \end{aligned}$$

Now,

$$P\left(\tilde{\theta} = \text{GREAT}|\tilde{y} = y\right) = \frac{P\left(Y = y\middle|\tilde{\theta} = \text{GREAT}\right) \cdot P\left(\tilde{\theta} = \text{GREAT}\right)}{P(\tilde{y} = y)} = \frac{(2^y/y!)e^{-2} \cdot 0.10}{P(\tilde{y} = y)}$$

$$P\left(\tilde{\theta} = \text{GOOD}|\tilde{y} = y\right) = \frac{P\left(\tilde{y} = y\middle|\tilde{\theta} = \text{GOOD}\right) \cdot P\left(\tilde{\theta} = \text{GOOD}\right)}{P(\tilde{y} = y)} = \frac{(1^y/y!)e^{-1} \cdot 0.50}{P(\tilde{y} = y)}$$

$$P\left(\tilde{\theta} = \text{POOR}|\tilde{y} = y\right) = \frac{P\left(\tilde{y} = y\middle|\tilde{\theta} = \text{POOR}\right) \cdot P\left(\tilde{\theta} = \text{POOR}\right)}{P(\tilde{y} = y)} = \frac{(0.5^y/y!)e^{-0.5} \cdot 0.40}{P(\tilde{y} = y)}$$

where $P(\tilde{y} = y) = P\left(\tilde{y} = y\middle|\tilde{\theta} = \text{GREAT}\right) \cdot P\left(\tilde{\theta} = \text{GREAT}\right) + P\left(\tilde{y} = y\middle|\tilde{\theta} = \text{GOOD}\right) \cdot P\left(\tilde{\theta} = \text{GOOD}\right)$
$+ P\left(\tilde{y} = y\middle|\tilde{\theta} = \text{POOR}\right) \cdot P\left(\tilde{\theta} = \text{POOR}\right)$

This gives

$$E(R(\text{"Hire"}|y)|y) =$$
$$= 60000 \cdot \frac{(2^y/y!)e^{-2} \cdot 0.10}{P(\tilde{y} = y)} + 15000 \cdot \frac{(1^y/y!)e^{-1} \cdot 0.50}{P(\tilde{y} = y)} - 30000 \cdot \frac{(0.5^y/y!)e^{-0.5} \cdot 0.40}{P(\tilde{y} = y)}$$
$$= \frac{6000 \cdot (2^y/y!)e^{-2} + 7500 \cdot (1^y/y!)e^{-1} - 12000 \cdot (0.5^y/y!)e^{-0.5}}{P(\tilde{y} = y)}$$

We now must investigate for which values of y the optimal action $a''|y$ is equal to "Hire" and for which y this action is equal to "Do not hire".

The optimal action is "Hire" when $E(R(\text{"Hire"}|y)|y) > E(R(\text{"Do not hire"}|y)|y) = 0$

$$\Leftrightarrow$$

$$\frac{6000 \cdot (2^y/y!)e^{-2} + 7500 \cdot (1^y/y!)e^{-1} - 12000 \cdot (0.5^y/y!)e^{-0.5}}{P(\tilde{y} = y)} > 0$$

$$\Leftrightarrow \langle \text{since } P(\tilde{y} = y) > 0 \rangle$$

$$6000 \cdot (2^y/y!)e^{-2} + 7500 \cdot (1^y/y!)e^{-1} - 12000 \cdot (0.5^y/y!)e^{-0.5} > 0$$

$$\Leftrightarrow \langle \text{since } y! > 0 \rangle$$

$$6000 \cdot 2^y \cdot e^{-2} + 7500 \cdot e^{-1} - 12000 \cdot 0.5^y \cdot e^{-0.5} > 0$$

$$\Leftrightarrow 6000 \cdot 2^y \cdot e^{-2} + 7500 \cdot e^{-1} - 12000 \cdot \frac{1}{2^y} \cdot e^{-0.5} > 0$$

$$\Leftrightarrow \langle \text{since } 2^y > 0 \rangle$$

$$6000 \cdot e^{-2} \cdot 2^{2y} + 7500 \cdot e^{-1} \cdot 2^y - 12000 \cdot e^{-0.5} > 0$$

$$\Leftrightarrow 2^{2y} + \frac{6000}{7500} e^1 \cdot 2^y - \frac{12000}{7500} e^{1.5} > 0$$

$$\Rightarrow \langle \text{since } 2^y > 0 \rangle$$

$$2^y > -\frac{6000}{2 \cdot 7500} e^1 + \sqrt{\left(-\frac{6000}{2 \cdot 7500} e^1\right)^2 + \frac{12000}{7500} e^{1.5}}$$

$$\Leftrightarrow 2^y > 1.802835 \Leftrightarrow y > \frac{\log(1.802835)}{\log(2)} \approx 0.85$$

Hence,

$$E(R(a"|y)|y) = \begin{cases} E(R(\text{"Hire"}|y)|y) & y \geq 1 \\ E(R(\text{"Do not hire"}|y)|y) & y = 0 \end{cases}$$

and

$$\mathrm{VSI}(y) = E(R(a"|y)|y) - E(R(a'|y)|y)$$

$$= \begin{cases} E(R(\text{"Hire"}|y)|y) - E(R(\text{"Hire"}|y)|y) = 0 & y \geq 1 \\ E(R(\text{"Do not hire"}|y=0)|y=0) - E(R(\text{"Hire"}|y=0)|y=0) & y = 0 \end{cases}$$

$$E(R(\text{"Do not hire"}|y=0)|y=0) = 0$$

$$E(R(\text{"Hire"}|y=0)||y=0)$$

$$= \frac{6000 \cdot (2^0/0!)e^{-2} + 7500 \cdot (1^0/0!)e^{-1} - 12000 \cdot (0.5^0/0!)e^{-0.5}}{P(\tilde{y}=0)}$$

$$= \frac{6000e^{-2} + 7500e^{-1} - 12000e^{-0.5}}{P(\tilde{y}=0)}$$

and thus

$$\mathrm{VSI}(y) = \begin{cases} 0 & y \geq 1 \\ -\dfrac{6000e^{-2} + 7500e^{-1} - 12000e^{-0.5}}{P(Y=0)} & y = 0 \end{cases}$$

…and finally,

$$EVSI = \sum_{y=0}^{\infty} \mathrm{VSI}(y) \cdot P(\tilde{y} = y) =$$

$$= \mathrm{VSI}(0) \cdot P(\tilde{y} = 0) + \cancel{\sum_{y=1}^{\infty} 0 \cdot P(\tilde{y} = y)} =$$

$$= \mathrm{VSI}(0) \cdot P(\tilde{y} = 0) =$$

$$= -\frac{6000e^{-2} + 7500e^{-1} - 12000e^{-0.5}}{\cancel{P(\tilde{y} = 0)}} \times \cancel{P(\tilde{y} = 0)}$$

$$\approx 3707$$

(c) Suppose that the owner of the dealership can purchase sample information at the rate of $10 per day by hiring the salesman on a temporary basis. He must sample in units of four days, however, for a salesman's work week consists of four days. He considers hiring the salesman for one week (four days). Find EVSI and ENGS for this proposed sample.

$$ENGS = EVSI - CS = 3707 - 4 \cdot 10 = 3667$$

Exercise 6.17

17. In Exercise 16, suppose that you also want to consider other sample sizes.
   (a) Find EVSI for a sample of size 2.
   (b) Find EVSI for a sample of size 5.
   (c) Find EVSI for a sample of size 10.
   (d) If the cost of sampling is \$0.50 per unit sampled, find the expected net gain of sampling (ENGS) for samples of sizes 1, 2, 5, and 10.

Exercise 6.16 was demonstrated at Meeting 11

DECISION	PROPORTION OF CUSTOMERS BUYING, θ				
	0.10	0.20	0.30	0.40	0.50
Stock 100	−10	−2	12	22	40
Stock 50	−4	6	12	16	16
Do not stock	0	0	0	0	0

16. In Exercise 15, suppose that sample information is available in the form of a random sample of consumers. For a sample of size *one*,
   (a) find the posterior distribution if the one person sampled will purchase the item, and find the value of this sample information;
   (b) find the posterior distribution if the one person sampled will *not* purchase the item, and find the value of this sample information;
   (c) find the expected value of sample information.

c) $\mathrm{EVSI}(1) = 3.2 \cdot 0.26 + 0 \cdot 0.74 = 0.832$

DECISION	PROPORTION OF CUSTOMERS BUYING, θ				
	0.10	0.20	0.30	0.40	0.50
Stock 100	–10	–2	12	22	40
Stock 50	–4	6	12	16	16
Do not stock	0	0	0	0	0

(a)

We need to consider all possible outcomes in a sample of size 2, i.e.

BUY, BUY
BUY, NOT BUY
NOT BUY, BUY
NOT BUY, NOT BUY

However, the second and third outcome are equal by symmetry

DECISION	PROPORTION OF CUSTOMERS BUYING, θ				
	0.10	0.20	0.30	0.40	0.50
Stock 100	–10	–2	12	22	40
Stock 50	–4	6	12	16	16
Do not stock	0	0	0	0	0

BUY, BUY:

Posterior distribution: $P(\theta|\text{BUY,BUY}) = \dfrac{P(\text{BUY,BUY}|\theta)\cdot P(\theta)}{P(\text{BUY,BUY})} = \dfrac{P(\text{BUY,BUY}|\theta)\cdot P(\theta)}{\sum_\lambda P(\text{BUY,BUY}|\lambda)\cdot P(\lambda)}$

$P(\text{BUY,BUY}|\theta) = \theta^2$

$\Rightarrow$

$P(\text{BUY,BUY})$

$= 0.10^2 \cdot 0.2 + 0.20^2 \cdot 0.3 + 0.30^2 \cdot 0.3 + 0.40^2 \cdot 0.1 + 0.50^2 \cdot 0.1 = 0.082$

$P(0.10|\text{BUY,BUY}) = 0.10^2 \cdot 0.2/0.082 \approx 0.0244$

$P(0.20|\text{BUY,BUY}) = 0.20^2 \cdot 0.3/0.082 \approx 0.1463$

$P(0.30|\text{BUY,BUY}) = 0.30^2 \cdot 0.3/0.082 \approx 0.3293$

$P(0.40|\text{BUY,BUY}) = 0.40^2 \cdot 0.1/0.082 \approx 0.1951$

$P(0.50|\text{BUY,BUY}) = 0.50^2 \cdot 0.1/0.082 \approx 0.3049$

$$\text{VSI(BUY,BUY)} = E(R(a''|\text{BUY,BUY})|\text{BUY,BUY}) - E(R(a')|\text{BUY,BUY})$$

$$a' = \langle \text{ from } \quad \text{exercise } \ 6.15\rangle = \text{Stock } \ 50$$

$$a''|\text{BUY,BUY} = \underset{a}{\text{argmax}}\{E(R(a)|\text{BUY,BUY})\}$$

$$E(R(a)|\text{BUY,BUY}) = \sum_{\theta} R(a,\theta) \cdot P(\theta|\text{BUY,BUY})$$

$$\Rightarrow$$

$$E(R(\text{Stock } 100)|\text{BUY,BUY})$$
$$= (-10) \cdot 0.0244 \ldots + (-2) \cdot 0.1463 \ldots + 12 \cdot 0.3293 \ldots +$$
$$22 \cdot 0.1951 \ldots + 40 \cdot 0.3049 \ldots \approx \boxed{19.90}$$
$$E(R(\text{Stock } 50)|\text{BUY,BUY}) = (-4) \cdot 0.0244 \ldots + 6 \cdot 0.1463 \ldots + 12 \cdot 0.3293 \ldots +$$
$$26 \cdot 0.1951 \ldots + 16 \cdot 0.3049 \ldots \approx 12.73$$
$$E(R(\text{Do not stock})|\text{BUY,BUY}) = 0 \cdot 0.0244 \ldots + 0 \cdot 0.1463 \ldots + 0 \cdot 0.3293 \ldots +$$
$$0 \cdot 0.1951 \ldots + 0 \cdot 0.3049 \ldots = 0$$

$$\Rightarrow a''|\text{BUY,BUY} = \text{Stock } \ 100$$
$$\Rightarrow \ \text{VSI(BUY,BUY)} =$$
$$= E(R(\text{Stock } 100|\text{BUY,BUY})|\text{BUY,BUY}) - E(R(\text{Stock } 50)|\text{BUY,BUY}) \approx$$
$$19.90 - 12.73 = 7.17$$

DECISION	PROPORTION OF CUSTOMERS BUYING, θ				
	0.10	0.20	0.30	0.40	0.50
Stock 100	–10	–2	12	22	40
Stock 50	–4	6	12	16	16
Do not stock	0	0	0	0	0

BUY, NOT BUY or NOT BUY, BUY:

Posterior distribution:

$$P(\theta|\text{BUY,NOT BUY}) = \frac{P(\text{BUY,NOT BUY}|\theta)\cdot P(\theta)}{P(\text{BUY,NOT BUY})} = \frac{P(\text{BUY,NOT BUY}|\theta)\cdot P(\theta)}{\sum_{\lambda} P(\text{BUY,NOT BUY}|\lambda)\cdot P(\lambda)}$$

$$P(\text{BUY,NOT BUY}|\theta) = \theta\cdot(1-\theta)$$

$\Rightarrow$

$$P(\text{BUY,NOT BUY}) = 0.10\cdot 0.90\cdot 0.2 + 0.20\cdot 0.80\cdot 0.3 + 0.30\cdot 0.70\cdot 0.3 + 0.40\cdot 0.50\cdot 0.1 + 0.50^2\cdot 0.1 = 0.178$$

$$P(0.10|\text{BUY,NOT BUY}) = 0.10\cdot 0.90\cdot 0.2/0.178 \approx 0.1011$$
$$P(0.20|\text{BUY,NOT BUY}) = 0.20\cdot 0.80\cdot 0.3/0.178 \approx 0.2697$$
$$P(0.30|\text{BUY,NOT BUY}) = 0.30\cdot 0.70\cdot 0.3/0.178 \approx 0.3539$$
$$P(0.40|\text{BUY,NOT BUY}) = 0.40\cdot 0.50\cdot 0.1/0.178 \approx 0.1348$$
$$P(0.50|\text{BUY,NOT BUY}) = 0.50^2\cdot 0.1/0.178 \approx 0.1404$$

$$\text{VSI}(\text{BUY,NOT BUY})$$
$$= E(R(a''|\text{BUY,NOT BUY})|\text{BUY,NOT BUY}) - E(R(a')|\text{BUY,NOT BUY})$$

$$a' = \text{Stock 50} \quad \text{(as before)}$$

$$a''|\text{BUY,NOT BUY} = \underset{a}{\text{argmax}}\{E(R(a)|\text{BUY,NOT BUY})\}$$

$$E(R(a)|\text{BUY,NOT BUY}) = \sum_{\theta} R(a,\theta) \cdot P(\theta|\text{BUY,NOT BUY})$$

$$\Rightarrow$$
$$E(R(\text{Stock 100})|\text{BUY,NOT BUY})$$
$$= (-10) \cdot 0.1011 \ldots + (-2) \cdot 0.2697 \ldots + 12 \cdot 0.3539 \ldots + 22 \cdot 0.1348 \ldots + 40 \cdot 0.1404 \ldots \approx \boxed{11.28}$$
$$E(R(\text{Stock 50})|\text{BUY,NOT BUY}) = (-4) \cdot 0.1011 \ldots + 6 \cdot 0.2697 \ldots + 12 \cdot 0.3539 \ldots + 26 \cdot 0.1348 \ldots + 16 \cdot 0.1404 \ldots \approx 9.87$$
$$E(R(\text{Do not stock})|\text{BUY,NOT BUY}) = 0 \cdot 0.1011 \ldots + 0 \cdot 0.2697 \ldots + 0 \cdot 0.3539 \ldots + 0 \cdot 0.1348 \ldots + 0 \cdot 0.1404 \ldots = 0$$

$$\Rightarrow a''|\text{BUY,NOT BUY} = \text{Stock 100}$$
$$\Rightarrow$$
$$\text{VSI}(\text{BUY,NOT BUY}) = E(R(\text{Stock 100})|\text{BUY,NOT BUY}) - E(R(\text{Stock 50})|\text{BUY,NOT BUY}) \approx 11.28 - 9.87 = 1.41$$

DECISION	PROPORTION OF CUSTOMERS BUYING, θ				
	0.10	0.20	0.30	0.40	0.50
Stock 100	–10	–2	12	22	40
Stock 50	–4	6	12	16	16
Do not stock	0	0	0	0	0

NOT BUY, NOT BUY:

Posterior distribution:

$$P(\theta|\text{NOT BUY,NOT BUY}) = \frac{P(\text{NOT BUY,NOT BUY}|\theta)\cdot P(\theta)}{P(\text{NOT BUY,NOT BUY})} = \frac{P(\text{NOT BUY,NOT BUY}|\theta)\cdot P(\theta)}{\sum_\lambda P(\text{NOT BUY,NOT BUY}|\lambda)\cdot P(\lambda)}$$

$$P(\text{NOT BUY,NOT BUY}|\theta) = (1-\theta)^2$$

$\Rightarrow$

$$P(\text{NOT BUY,NOT BUY}) = 0.90^2\cdot 0.2 + 0.80^2\cdot 0.3 + 0.70^2\cdot 0.3 + 0.60^2\cdot 0.1 + 0.50^2\cdot 0.1 = 0.562$$

$$P(0.10|\text{NOT BUY,NOT BUY}) = 0.90^2\cdot 0.2/0.562 \approx 0.2883$$
$$P(0.20|\text{NOT BUY,NOT BUY}) = 0.80^2\cdot 0.3/0.562 \approx 0.3416$$
$$P(0.30|\text{NOT BUY,NOT BUY}) = 0.70^2\cdot 0.3/0.562 \approx 0.2616$$
$$P(0.40|\text{NOT BUY,NOT BUY}) = 0.60^2\cdot 0.1/0.562 \approx 0.0641$$
$$P(0.50|\text{NOT BUY,NOT BUY}) = 0.50^2\cdot 0.1/0.562 \approx 0.0445$$

$\text{VSI}(\text{NOT BUY,NOT BUY}) =$
$= E(R(a''|\text{NOT BUY,NOT BUY})|\text{NOT BUY,NOT BUY}) - E(R(a')|\text{NOT BUY,NOT BUY})$

$a' = \text{Stock } 50$ (as before)

$$a''|\text{NOT BUY,NOT BUY} = \underset{a}{\operatorname{argmax}}\{E(a|\text{NOT BUY,NOT BUY})\}$$

$$E(a|\text{NOT BUY,NOT BUY}) = \sum_{\theta} R(a,\theta)\cdot P(\theta|\text{NOT BUY,NOT BUY})$$

$\Rightarrow$

$E(\text{Stock 100}|\text{NOT BUY,NOT BUY})$
$= (-10)\cdot 0.2883\ldots + (-2)\cdot 0.3416\ldots + 12\cdot 0.2616\ldots +$
$22\cdot 0.0641\ldots + 40\cdot 0.0445\ldots \approx 2.76$

$E(\text{Stock 50}|\text{NOT BUY,NOT BUY})$
$= (-4)\cdot 0.2883\ldots + 6\cdot 0.3416\ldots + 12\cdot 0.2616\ldots +$
$26\cdot 0.0641\ldots + 16\cdot 0.0445\ldots \approx \boxed{5.77}$

$E(\text{Do not stoc}|\text{NOT BUY,NOT BUY}) = 0\cdot 0.2883\ldots + 0\cdot 0.3416\ldots + 0\cdot 0.2616\ldots +$
$0\cdot 0.0641\ldots + 0\cdot 0.0445\ldots = 0$

$\Rightarrow a''|\text{NOT BUY,NOT BUY} = \text{Stock } 50$

$\Rightarrow$

$\text{VSI}(\text{NOT BUY,NOT BUY}) = E(\text{Stock 50}|\text{NOT BUY,NOT BUY}) -$
$E(\text{Stock 50}|\text{NOT BUY,NOT BUY}) = (5.77 - 5.77) = 0$

$$\mathrm{EVSI} = \sum_{y} \mathrm{VSI}(y)P(y) =$$

$$= \mathrm{VSI}(\mathrm{BUY,BUY}) \cdot P(\mathrm{BUY,BUY}) + 2 \cdot \mathrm{VSI}(\mathrm{BUY,NOT\ \ BUY}) \cdot P(\mathrm{BUY,NOT\ \ BUY})$$
$$+$$
$$\mathrm{VSI}(\mathrm{NOT\ \ BUY,NOT\ \ BUY}) \cdot P(\mathrm{NOT\ \ BUY,NOT\ \ BUY}) \ =$$

$$= 7.17 \cdot 0.082 + 2 \cdot 1.41 \cdot 0.178 + 0 \cdot 0.652 \approx 1.09$$

(b,c) Tedious to sort out the calculations for sample sizes greater than 2.

Use the fact that the sample outcome is that of binomial sampling:

$$P(\theta|\text{Sample outcome}) \propto P(\text{Sample outcome}|\theta) \times P(\theta) =$$
$$= P(y|\theta) \times P(\theta) = \binom{n}{y} \theta^{y} \cdot (1-\theta)^{n-y} \times P(\theta)$$

Let

$\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)^{\mathbf{T}}$ *column matrix*

$P(\boldsymbol{\theta}) = \left(P(\theta_1), P(\theta_2), P(\theta_3), P(\theta_4), P(\theta_5)\right)^{\mathrm{T}}$ *column matrix*

$P(y|\boldsymbol{\theta}) = \left(P\left(y|\theta_1\right), P\left(y|\theta_2\right), P\left(y|\theta_3\right), P\left(y|\theta_4\right), P\left(y|\theta_5\right)\right)^{\mathrm{T}}$ *column matrix*

$$\Rightarrow P(\boldsymbol{\theta}|y) = \frac{P(y|\boldsymbol{\theta}) \odot P(\boldsymbol{\theta})}{P(y|\boldsymbol{\theta})^{\mathrm{T}} \cdot P(\boldsymbol{\theta})} = \frac{\boldsymbol{\theta}^{\circ y} \odot (1-\boldsymbol{\theta})^{\circ(n-y)} \odot P(\boldsymbol{\theta})}{[\boldsymbol{\theta}^{\circ y} \odot (1-\boldsymbol{\theta})^{\circ(n-y)}]^{\mathrm{T}} \cdot P(\boldsymbol{\theta})}$$ *column matrix*

$\odot$ elementwise multiplication ; $\ldots^{\circ\cdots}$ elementwise exponentiation

$\boldsymbol{ER}(y) = \left(E(R(\text{Stock 100})|y), E(R(\text{Stock 50})|y), E(R(\text{Do not stock})|y)\right)^{\mathrm{T}}$ *column matrix*

$$\boldsymbol{Rmat} = \begin{pmatrix} -10 & -2 & 12 & 22 & 40 \\ -4 & 6 & 12 & 16 & 16 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ *payoff table as a 3×5 matrix*

$$\Longrightarrow \boldsymbol{ER}(y) = \boldsymbol{Rmat} \cdot P(\boldsymbol{\theta}|y)$$

(b)

y	θ	$P(\theta)$	$P(y \mid \theta)$	$P(\theta \mid y)$	E(R(Stock *100*)|y)	E(R(Stock 50)|y)	E(R(Do not stock)|y)	a''	a'	E((a')|y)	VSI	$P(y)$	EVSI
0	0.1	0.2	0.591	0.425									
	0.2	0.3	0.328	0.354									
	0.3	0.3	0.168	0.182									
	0.4	0.1	0.078	0.028									
	0.5	0.1	0.031	0.011	-1.716	3.230	0	Stock 50	Stock 50	3.230	0	0.2777	*0*
1	0.1	0.2	0.328	0.194									
	0.2	0.3	0.410	0.364									
	0.3	0.3	0.360	0.320									
	0.4	0.1	0.259	0.077									
	0.5	0.1	0.156	0.046	4.703	7.206	0	Stock 50	Stock 50	7.206	0	0.3381	*+0*

2	0.1	0.2	0.073	0.062									
	0.2	0.3	0.205	0.262									
	0.3	0.3	0.309	0.395									
	0.4	0.1	0.346	0.147									
	0.5	0.1	0.313	0.133	12.169	10.555	0	Stock 100	Stock 50	10.555	1.6139	0.2344	*+0.38*
3	0.1	0.2	0.008	0.015									
	0.2	0.3	0.051	0.138									
	0.3	0.3	0.132	0.358									
	0.4	0.1	0.230	0.208									
	0.5	0.1	0.313	0.282	19.703	12.893	0	Stock 100	Stock 50	12.893	6.8101	0.1110	*+0.76*
4	0.1	0.2	5e-04	0.003									
	0.2	0.3	0.006	0.057									
	0.3	0.3	0.028	0.252									
	0.4	0.1	0.077	0.227									
	0.5	0.1	0.156	0.462	26.354	14.373	0	Stock 100	Stock 50	14.373	11.9804	0.0338	*+0.40*
5	0.1	0.2	0	4e-04									
	0.2	0.3	3e-04	0.019									
	0.3	0.3	0.002	0.147									
	0.4	0.1	0.010	0.206									
	0.5	0.1	0.031	0.628	31.363	15.213	0	Stock 100	Stock 50	15.213	16.1503	0.0050	***+0.08***
													= 1.62

(c)

y	θ	$P(\theta)$	$P(y \mid \theta)$	$P(\theta \mid y)$	$E(R(\text{Stock } 100) \mid y)$	$E(R(\text{Stock } 50) \mid y)$	$E(R(\text{Do not stock}) \mid y)$	a''	a'	$E((a') \mid y)$	VSI	$P(y)$	EVSI
0	0.1	0.2	0.349	0.628									
	0.2	0.3	0.107	0.290									
	0.3	0.3	0.028	0.076									
	0.4	0.1	0.006	0.005									
	0.5	0.1	0.001	9e-04	-5.785	0.245	0	Stock 50	Stock 50	0.245	0	0.1111	*0*
1	0.1	0.2	0.387	0.389									
	0.2	0.3	0.268	0.404									
	0.3	0.3	0.121	0.182									
	0.4	0.1	0.040	0.020									
	0.5	0.1	0.010	0.005	-1.868	3.457	0	Stock 50	Stock 50	3.457	0	0.1993	*+0*

2	0.1	0.2	0.194	0.180									
	0.2	0.3	0.302	0.420									
	0.3	0.3	0.234	0.325									
	0.4	0.1	0.121	0.056									
	0.5	0.1	0.044	0.020	3.306	6.916	0	Stock 50	Stock 50	6.916	0	0.2159	*+0*
3	0.1	0.2	0.057	0.062									
	0.2	0.3	0.201	0.326									
	0.3	0.3	0.267	0.432									
	0.4	0.1	0.215	0.116									
	0.5	0.1	0.117	0.063	9.002	9.768	0	Stock 50	Stock 50	9.768	0	0.1851	*+0*
4	0.1	0.2	0.011	0.017									
	0.2	0.3	0.088	0.197									
	0.3	0.3	0.200	0.447									
	0.4	0.1	0.251	0.187									
	0.5	0.1	0.205	0.153	15.024	11.911	0	Stock 100	Stock 50	11.911	3.1121	0.1343	*+0.42*
5	0.1	0.2	0.002	0.004									
	0.2	0.3	0.026	0.095									
	0.3	0.3	0.103	0.369									
	0.4	0.1	0.201	0.240									
	0.5	0.1	0.246	0.294	21.217	13.509	0	Stock 100	Stock 50	13.509	7.7089	0.0838	*+0.65*

6	0.1	0.2	1e-04	6e-04									
	0.2	0.3	0.006	0.037									
	0.3	0.3	0.037	0.249									
	0.4	0.1	0.112	0.251									
	0.5	0.1	0.205	0.462	26.922	14.621	0	Stock 100	Stock 50	14.621	12.3011	0.0444	*+0.55*
7	0.1	0.2	0.0000	1e-04									
	0.2	0.3	8e-04	0.013									
	0.3	0.3	0.009	0.143									
	0.4	0.1	0.043	0.225									
	0.5	0.1	0.117	0.620	31.428	15.302	0	Stock 100	Stock 50	15.302	16.1256	0.0189	*+0.30*
8	0.1	0.2	0.0000	0.000									
	0.2	0.3	1e-04	0.004									
	0.3	0.3	0.001	0.073									
	0.4	0.1	0.011	0.180									
	0.5	0.1	0.044	0.743	34.555	15.669	0	Stock 100	Stock 50	15.669	18.8859	0.0059	*+0.11*

9	0.1	0.2	0.0000	0.000									
	0.2	0.3	0.0000	0.001									
	0.3	0.3	1e-04	0.035									
	0.4	0.1	0.002	0.134									
	0.5	0.1	0.010	0.830	36.566	15.849	0	Stock 100	Stock 50	15.849	20.7167	0.0012	*+0.02*
10	0.1	0.2	0.0000	0.0000									
	0.2	0.3	0.0000	3e-04									
	0.3	0.3	0.0000	0.016									
	0.4	0.1	1e-04	0.095									
	0.5	0.1	0.001	0.888	37.820	15.933	0	Stock 100	Stock 50	15.933	21.8876	0.0001	*+0.002*
													= 2.05

(d) $n = 1$: ENGS(1) = EVSI(1) – CS(1) ≈ 0.832 – 0.50·1 = 0.332

$n = 2$: ENGS(2) = EVSI(2) – CS(2) ≈ 1.09 – 0.50·2 = 0.09

$n = 5$: ENGS(5) = EVSI(5) – CS(5) ≈ 1.62 – 0.50·5 = –0.88

$n = 10$: ENGS(10) = EVSI(10) – CS(10) ≈ 2.05 – 0.50·10 = –2.95