Master's Thesis Proposal

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1 Lower Bounds for Matrix Multiplication

In the late 1960s, Strassen [5] discovered an algorithm for multiplying 2×2-matrices using only 7 essential multiplications instead of 8. He then used this algorithm recursively to derive an algorithm for multiplying $n \times n$-matrices with $\mathcal{O}(n^{\log_2 7}) = \mathcal{O}(n^{2.808})$ arithmetic operations. Since then, a lot of effort has been spent on improving Strassen's upper bound and the current "world record" is held by Coppersmith and Winograd [2]. They exhibit an algorithm with $\mathcal{O}(n^{2.376})$ arithmetic operations. However, the constants hidden in the $\mathcal{O}$-notation are far to huge to make this algorithm usable in practise. This is also the case with most other algorithms, except Strassen's algorithm.

One way to obtain faster algorithms of practical relevance is to find a good algorithm for multiplying matrices of some small format. Since any bilinear algorithm for multiplying 2×2-matrices requires at least 7 essential multiplications [6], we have to look for another format. The best bilinear algorithm for multiplying 3×3-matrices known so far uses 23 essential multiplications [4]. This yields an algorithm for multiplying $n \times n$-matrices with $\mathcal{O}(n^{\log_3 23}) = \mathcal{O}(n^{2.858})$ arithmetic operations. To improve Strassen's algorithm, an algorithm with 21 or less essential bilinear multiplications is required. The 3×3 format is of particular interest, since it is the first one for which the best lower (At least 19 multiplications are required [1].) and upper bounds known so far differ significantly.

The goal of the proposed thesis project would be to try to find a better lower bound for 3×3-matrix multiplication. Johnson and McLoughlin [3] present further bilinear algorithms for 3×3 matrices using 23 essential multiplications that are not equivalent to Laderman's approach. More precisely, they specify rational coefficients A_{ij}^r, B_{kl}^r, and C_{mn}^r such that, with $N = 3$ and $M = 23$, the equation

$$\sum_{p=1}^{N} X_{np} Y_{pm} = \sum_{r=1}^{M} \left(\sum_{i,j=1}^{N} A_{ij}^r X_{ij} \right) \left(\sum_{k,l=1}^{N} B_{kl}^r Y_{kl} \right) C_{mn}^r \tag{1}$$

is an identity for $N \times N$-matrices X and Y.

A necessary and sufficient condition for (1) to hold identically in X and Y is that the coefficients satisfy

$$\sum_{r=1}^{M} A^r_{ij} B^r_{kl} C^r_{mn} = \delta_{mi}\delta_{jk}\delta_{ln}. \tag{2}$$

This system of polynomial equations is where the thesis project is proposed to make its attack on trying to prove better lower bounds for 3×3-matrix multiplication. Namely, if one could show that the above system does not have any solution for $N = 3$ and some $M \in \{20, 21, 22\}$, this would be a proof of a new lower bound. This attack would rely on the hope of low degree of the Hilbert's Nullstellensatz certificates for the above polynomial systems and on large-scale linear-algebra computation. It seems plausible that this project would even need to use computing resources from the National Supercomputer Centre in Linköping.

1.1 Contact

Does this sound like an interesting project? Would you like to know more about it? Do not hesitate to contact the author!

References

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