Complexity and Approximability of Problems Related to Integer Programming

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1 Optimisation and integer programming

Many problems of both practical and theoretical importance concern themselves not only with finding a solution but also finding a 'best' solution in some sense. As a concrete example, let us consider the travelling salesperson problem (TSP): a salesperson has to visit some specified towns on a tour starting and ending in her/his home town; in which order should the towns be visited in order to minimize the length of the tour? We see that this problem actually becomes meaningless if we only ask the question whether a tour exists or not (under some plausible assumptions on the available communication system). This kind of problems are often referred to as *combinatorial optimization problems*. They are typically computationally hard (**NP**-hard) to solve exactly but they can sometimes be efficiently approximated. Let us consider TSP once again: if the distances satisfy the triangle inequality, then the exact problem is **NP**-hard but a tour that is at most 1.5 longer than the optimal tour can be found in low-order polynomial time. If the distances are Euclidean, then the exact problem is still **NP**-hard but the optimal tour can be approximated within $1+\epsilon$ for every $\epsilon > 0$, cf. [1]. Thus, useful results can indeed be obtained by using approximate methods.

In this project, we concentrate on optimisation problems that are (more or less) closely related to *integer programming*, i.e. the optimisation problem $\max\{c^T x \mid Ax \leq b, x \in \mathbb{N}^n\}$ where A is an $m \times n$ rational matrix, b is a rational m-vector, and c is a rational n-vector. Integer programming has been used for solving an enormous amount of problems ranging from network planning to optimal control to scheduling to coding theory, and it is an archetypical example of an **NP**-hard problem. This problem has been intensively studied from various viewpoints and a good overview of this research can be found in [9].

2 Tractable cases of integer programming

One of our current research projects is to answer the following question: *for which restrictions on the types of allowed inequalities is the integer program-*

ming problem tractable? Hence, we are asking for which restrictions on the rows of the constraint matrix (A, b) that guarantees that an optimal solution can be found in polynomial time.

Our initial investigations [2] indicates that if the variables are allowed to take values from an infinite domain (i.e., $\mathbb{N}$), then the tractable classes, with respect to restrictions on the rows of the constraint matrix, are very restricted. Fortunately, in most practical applications the variables can be assumed to take their values from a finite domain (e.g., the distance between any two cities in the world is finite, the profit of selling item X is finite, the cost of building factory Y is finite, and so on). Integer programming restricted to finite domains is sometimes called bounded integer programming, and from now on, when we speak of integer programming we mean bounded integer programming.

Although most known tractable classes for integer programming are defined in terms of simultaneous restrictions on the types of allowed inequalities and how they are allowed to be connected, there is one well known tractable class which is defined solely in terms of restrictions on the types of allowed inequalities: the monotone inequalities due to Hochbaum and Naor [3]. An inequality is monotone if it contains at most one positive variable and at most one negative variable, i.e., if it is of the form $a_1x - a_2y \leq c$. In terms of restrictions on the rows of the constraint matrix, Hochbaum's and Naor's result say that if every row of A contains at most one positive and at most one negative component, then the integer programming problem is tractable (but it becomes NP-hard if infinite domains are allowed).

Our preliminary studies show that there are several more tractable classes of integer programming with restrictions on the types of allowed inequalities. For example, one of these tractable classes explain the tractability of the critical independent set problem [10] and the $2w$-stable set problem [8]. One topic for a master's thesis could be to try to extend these tractable classes and/or to find new tractable classes for integer programming with restrictions on the types of allowed inequalities.

3 The Max Sol problem

Modelling real-world problems as integer programming is occasionally extremely difficult. The reason for this is (at least partly) that one can only express linear inequalities with the constraint matrix A, and this makes it cumbersome to express certain relations. A plausible alternative is to use the Max Sol problem instead; here, the objective is to maximize a linear function over a set of relations that is not restricted to only contain linear relations. This makes modelling much simpler but it sometimes makes it more difficult to solve the problem exactly, too. This implies that it is very important to find useful approximation algorithms for Max Sol when restricted to different sets of allowed relations.

Many combinatorial problems (in particular so-called *constraint satisfaction problems*) can be studied by exploiting powerful methods taken from clone theory and universal algebra, and this approach has led to a number of very strong results. The problem Max Sol can be studied by using the very same methods and this has resulted in a large number of results [4–7], too. These papers consider restricted classes of the problem and completely characterise the complexity and/or approximability of each class. The obvious way of continuing this work is to use the algebraic approach for studying other cases of Max Sol, and many different master's thesis projects can be identified within this framework.

4 Contact

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