Inductively Defined Sets and Recursive Functions

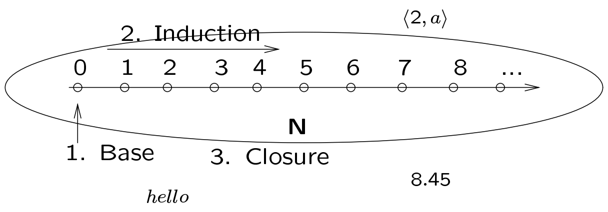
Countable sets can be define inductively.

- $0 \in \mathbf{N}$.
- If $x \in \mathbf{N}$ then $S(x) \in \mathbf{N}$.
- Nothing else is in $\mathbf{N}$.

Principle of Inductive Set Definitions

- **Base:** Define some fundamental or "first" elements in the set, e.g. "$0 \in \mathbf{N}$".
- **Induction:** Describe how the rest of the elements can be constructed from the previously defined, e.g. "if $x \in \mathbf{N}$ then $S(x) \in \mathbf{N}$".
- **Closure:** Establish that nothing else is in the set.

Principle of Inductive Set Definitions



Example: Strings

An *alphabeth* is a set of letters, e.g.

$A = \{$ a, b, c, d,..., x, y, z $\}$

We now define the set of *strings* (finite sequences of letters) of A, $Str(A)$. The empty string (the string of length 0) is denoted ϵ.

- **Base:** $\epsilon \in Str(A)$.
- **Induction:** If $\alpha \in Str(A)$ and $y \in A$, then $y\alpha \in Str(A)$.
- **Closure:** Nothing else is in $Str(A)$.

Note: $Str(A)$ is an infinite set, but all strings in $Str(A)$ are finite.

We can use inductive definition of $Str(A)$ to generate new members of the set.

- **Base:** $\epsilon \in Str(A)$.

So we have the following element in $Str(A)$:
ϵ (the empty string).

- **Induction:** If $\alpha \in Str(A)$ and $y \in A$, then $y\alpha \in Str(A)$.

So , the following elements also belong to $Str(A)$:
a b c d e f g h i j k l m n o p q r s t u v x y z

Which gives the following elements in $Str(A)$, by induction:
aa ab ac ad ae af ag ... zp zq zr zs zt zu zv zx zy zz

As well as:
aaa aab aac aad aae aaf ... zzs zzt zzu zzv zzx zzy zzz

And so on...

- **Closure:** Nothing else is in $Str(A)$.

The following elements does *not* belong to $Str(A)$:

aaaa45 $\{\langle$ab, ba$\rangle, Olle\}$ $S(S(S(0)))$ 12.563 ♠

Recursive Functions

We will now construct a function that takes a string as an argument and maps it on the backward-version of the string.

$$Backw : Str(A) \to Str(A)$$

Example: $Backw$(axiomatization) = noitazitamoixa.

Problem: The domain of $Backw$, $Str(A)$, is infinite, so we can not write all tuples in $Backw$.

$$Backw = \{ \langle \epsilon, \epsilon \rangle, \langle \text{a, a} \rangle, \langle \text{b, b} \rangle, \langle \text{c, c} \rangle, \langle \text{e, e} \rangle, \langle \text{f, f} \rangle, \ldots \}$$

But, we can use the way we defined $Str(A)$ to get a finite description of $Backw$!

Recursive Definition of $Backw$

- **Base:** $Backw(\epsilon) = \epsilon$.
- **Induction:** $Backw(y\alpha) = Backw(\alpha)y$ if $y \in A$ and $\alpha \in Str(A)$.

Example: $Backw(\text{hej}) = \underline{Backw(\text{ej})}\text{h} = \underline{\underline{Backw(\text{j})}}\text{eh} = \underline{\underline{Backw(\epsilon)}}\text{jeh} = \epsilon\text{jeh} = \text{jeh}$.

Comparison: Inductive Definition and Recursive Function

Inductive Definition
Base: $\epsilon \in Str(A)$.
Induction: If $y \in A$ and $\alpha \in Str(A)$ then $y\alpha \in Str(A)$.
Closure: Nothing else is in $Str(A)$.

Recursive Function
Base: $Backw(\epsilon) = \epsilon$.
Induction: $Backw(y\alpha) = Backw(\alpha)y$ if $y \in A$ and $\alpha \in Str(A)$.

Rerun — $+ : (N \times N) \to N$ as a recursive function

- **Base:** $x + 0 = x$.
- **Induction:** $x + S(y) = S(x) + y$ if $x, y \in N$.

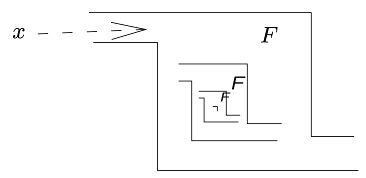
Example (1 + 2 again): $S(0) + S(S(0)) = S(S(0)) + S(0) = S(S(S(0))) + 0 = \underbrace{S(S(S(0)))}_{=3}$

Principle of Recursive Functions

$F : A \to B$ (A and B are often equal)

- **Base:** For every base element $b \in A$, define the value of $F(b)$.
- **Induction:** For every inductively defined element $i \in A$, define the value of $F(i)$ in terms of F itself and other previously defined functions and values in A.

The Paradox of a Function That Is Defined in Terms of Itself



The fact that we define a function in terms of itself can appear suspicious; Does this not lead to an infinite loop?

No, not if we do it properly. The point is that every time the function "uses itself", it does so on a "smaller" number of elements than before. Sooner or later we will reach a base element (e.g. 0 for the natural numbers), which will end the recursion.

Also note that breaking down the elements into smaller pieces is done according to the inductive definition of the set of elements. It is possible to view this as a backward run of the inductive definition.

Popular Example: Faculty Function

$Fac : N \rightarrow N$ is a function, such that
$Fac(n) = 1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n$.

- **Base:** $Fac(0) = 1$.
- **Induction:** $Fac(n) = Fac(n-1) \cdot n$ for $n > 0$.

Faculty Function Cont'd

$$
\begin{array}{llll}
Fac(3) & = 6 & & \\
Fac(2)\cdot 3 & = 2\cdot 3 & = 6 & \\
Fac(1)\cdot 2 & = 1\cdot 2 & = 2 & \\
Fac(0)\cdot 1 & = 1\cdot 1 & = 1 & \\
1 & & &
\end{array}
$$

Example (Morse Translator):

$$
\begin{aligned}
Morse &: Alpha \rightarrow Code && (1)\\
Alpha &= \{\text{a}, \text{b}, \text{c}, \ldots\} && (2)\\
Code &= \{\cdot, -, \cdot\cdot, \cdot-, -\cdot, --, \ldots\} && (3)\\
Morse &= \{\langle \text{a}, \cdot-\rangle, \ldots\} && (4)\\
MorseStr &: Str(A) \rightarrow Str(Mo) && (5)
\end{aligned}
$$

Base: $MorseStr(\epsilon) = \epsilon$ (6)

Ind: $MorseStr(x\alpha) = Morse(x)MorseStr(\alpha)$ (7)

Example (Palindrome):

$$
\begin{aligned}
Palindrome &: Str(A) \rightarrow B && (8)\\
eq &: (A \times A) \rightarrow B && (9)\\
eq(x,y) &= 1 \text{ if } x = y, \text{ else } 0. && (10)
\end{aligned}
$$

Base: $Palindrome(\epsilon) = 1$ (11)

$Palindrome(x) = 1, \; x \in A$ (12)

Ind: $Palindrome(x\alpha y) = eq(x,y) \wedge Palindrom(\alpha),$ (13)

$x \in A, y \in A$ and $\alpha \in Str(A)$. (14)

Example (Ackermann's Function):

$$
\begin{aligned}
Ack &: (\mathbf{N} \times \mathbf{N}) \rightarrow \mathbf{N} && (15)\\
\textbf{Base:}\; Ack(0,n) &= n+1, \quad n \geq 0 && (16)\\
\textbf{Ind:}\; Ack(m,0) &= Ack(m-1,1), \quad m \geq 0 && (17)\\
Ack(m,n) &= Ack(m-1, Ack(m,n-1)), \quad m,n && (18)
\end{aligned}
$$

Grows extremely fast. E.g. $Ack(4,2) >>$ the number of particles in the universe.

Induction Proofs

Every $x \geq 2$ where $x \in N$ is a product of one or more prime numbers.

How do we prove this?

A prime number p is a natural number that is not divisible by any other number than 1 and itself, and, therefore, cannot be written as $p = p_1 \cdot p_2$ for two other natural numbers $p_1,\ p_2$.

Proof

Let

$M = \{\ x\ | x \geq 2$ and x is a product of prime numbers. $\}$.

Base: 2 is a prime number, so $2 \in M$.

Induction Hypothesis: Assume that $x \in M$ holds, for all $x \leq n$.

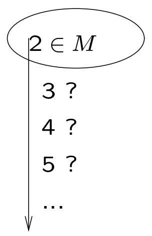
Induction Step: If $(n+1)$ is a prime number, then $(n+1) \in M$. If $(n+1)$ is not a prime number $(n+1)$ has to be the product of two numbers n_1 and n_2 (according to the definition of prime numbers), where $n_1 < (n+1)$ and $n_2 < (n+1)$.

n_1 and n_2 are products of prime numbers, according to the induction hypothesis , and since $(n+1) = n_1 \cdot n_2$, $(n+1)$ must be a product of prime numbers: $(n+1) \in M$.

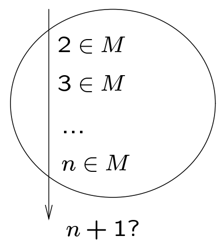
Thus, all $x \geq 2$ with $x \in N$ are products of prime numbers!

What Happened?

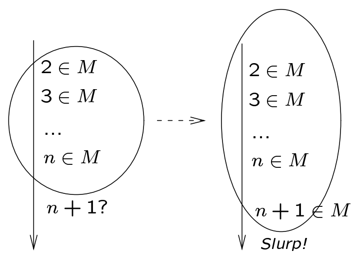
Base: 2 is a prime, so $2 \in M$.



Induction Hypothesis: Assume that $x \in M$ holds, for all $x \leq n$.



Induction Step: If $(n+1)$ is a prime number, then $(n+1) \in M$. If $(n+1)$ is not a prime number $(n+1)$ has to be the product of two numbers n_1 and n_2 (according to the definition of prime numbers), where $n_1 < (n+1)$ and $n_2 < (n+1)$.
n_1 and n_2 are products of prime numbers, according to the induction hypothesis , and since $(n+1) = n_1 \cdot n_2$, $(n+1)$ must be a product of prime numbers: $(n+1) \in M$.



Thus, all $x \geq 2$ with $x \in N$ are products of prime numbers! (x is choosen arbitrarily!)

$2 \in M$
$3 \in M$
$\dots$
$n \in M$
$n+1 \in M$
$\dots$

Practical Use of Induction Proofs

American Policy for Central America: The Domino Doctrine.

Base: Nicaragua will fall for communism.
Induction Hypothesis: Assume that a neighbour of country X falls.
Induction Step: If a neighbour of X falls, X itself will fall by communistic infiltration.

Conclusion: All countries in Central America will fall for communism. (Problem: the countries are not ordered)

A Better Example: Real Dominos on a Row

Base: The first domino falls.
Induction Hypothesis: Assume that the nth domino falls.
Induction Step: If the nth domino falls, the domino immediately after the nth (the $n+1$th domino) will fall too, since the nth domino will fall on it.

Conclusion: All dominos will fall.

Principle of Inductive Proofs

We want to show that $P(x)$ holds for all x (e.g. $P(x) = x \in \{\ x\ |x \geq 2$ and x is a product of primes. $\}$).

- **Base:** Show that $P(b)$ holds for all base elements b, (e.g. $P(0)$).
- **Induction Hypothesis:** Assume that $P(n)$ holds (or, $P(x)$ for all x such that $b \leq x \leq n$).
- **Induction Step:** Show, given the induction hypothesis, that $P(n+1)$ holds too.

Then we can draw the conclusion that $P(x)$ holds for all x such that $b \leq x$.

$M = \{\ x\ |x \geq 2$ and x is a product of prime numbers. $\}$.
Base: 2 is a prime number, so $2 \in M$.
Induction Hypothesis: Assume that $x \in M$ holds, for all $x \leq n$.
Induction Step: If $(n+1)$ is a prime number, then $(n+1) \in M$. If $(n+1)$ is not a prime number $(n+1)$ has to be the product of two numbers n_1 and n_2 (according to the definition of prime numbers), where $n_1 < (n+1)$ and $n_2 < (n+1)$.
n_1 and n_2 are products of prime numbers, according to the induction hypothesis , and since $(n+1) = n_1 \cdot n_2$, $(n+1)$ must be a product of prime numbers: $(n+1) \in M$.

Another Example

Show that $n \leq n!$ (n faculty) for all natural numbers $n \geq 0$.

That is, $P(n) = (n \leq n!)$.

Proof

Base: $0 \leq 1 = 0!$ and $1 \leq 1 = 1!$.

Induction Hypothesis: Assume that $n \leq n!$ for a natural number $n \geq 1$.

Induction Step: Since we have assumed that $n \leq n!$ we know that $n+1 \leq n!+1$. Furthermore:
$n!+1 \ \leq\ (n! \cdot n)+1 \ \leq\ (n! \cdot n)+(n! \cdot 1) \ =\ n! \cdot (n+1) \ =\ (n+1)!$. Thus: $n+1 \leq (n+1)!$.
We have showed that if $n \leq n!$ then $n+1 \leq (n+1)!$.

Conclusion: For all natural numbers $n \geq 0$, $n \leq n!$ holds.

Important questions:

- What do I want to do induction over?
- Which element(s) should I use as base element(s)?
- Given an element n, how can I generate the next element $S(n)$ (there can be more than one way)?

- Inductive Definition:
  Base: $b \in M$. Induction: if $b \in M$ then $S(b) \in M$.
  Closure: Nothing else is in M.
- Recursive Function:
  Base: $f(b) = c$. Induction: $f(S(c)) = f(c) \oplus ....$
- Induction Proof:
  Base: Show$P(b)$.
  Induction Hypothesis: Assume $P(x)$ for an arbitrary x.
  Induction Step: Show that if $P(x)$ holds, then $P(S(x))$ holds too.
  Conclusion: $P(x)$ holds for all $x \geq b$.