Finite and Infinite Sets

Finite Sets:

{a, b, c, d, . . . , , , }

$\{Pelle, Olle, Stina, Karo\}$

$\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,$
$15,16,17,18,19,20,21,22,23,24,25,26\}$

$\{x \mid x$ is an atom in the universe$\}$

Infinite Sets:

$\{x \mid x$ is a natural number$\}$

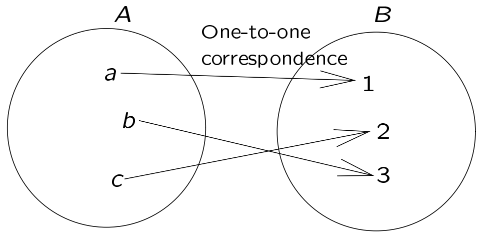
$\{x \mid x$ is an acceptable English sentence$\}$

Q

- When are two sets equally large (are equivalent)?
- How do we describe the size of a set?
- What does it mean that a set is infinite?

Equivalent Sets

Two sets A and B are *equivalent* iff there exists a one-to-one correspondence between them. We write $A \sim B$.



Equivalent Sets Cont'd

Recall: A one-to-one correspondence pairs the elements the elements in A and B .two and two.

Every element in A with exactly one element in B and *vice versa*.

Therefore: One-to-one correspondence $\Rightarrow$ equally many elements.

Cardinality

A measure on how many elements a set has.
Written $|A|$ for a set A.

For finite sets: Simply count the number of elements.

Example (Cardinality):

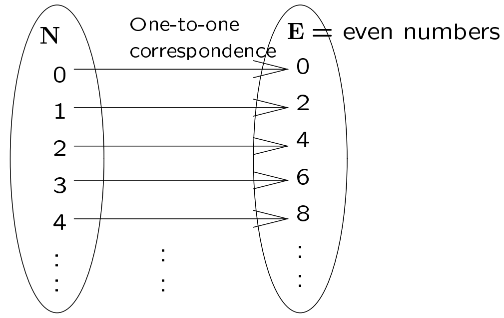
$|\{a,b,c\}| = 3$

Cardinality Cont'd

What cardinality does the set of natural numbers, $\mathbf{N} = \{0,1,2,\ldots\}$, have?

Count the elements:
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, ...

How many natural numbers are there? A mystery:



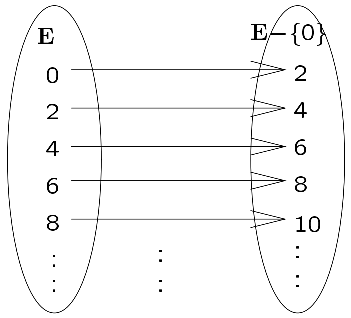
$\mathbf{N} \sim \mathbf{E}$, i.e. are equally large! But: $\mathbf{E} \subset \mathbf{N}$!!???

Definition A set if *infinite* iff it is equivalent to a proper subset of itself.

Theorem The set of natural numbers, $\mathbf{N}$, is infinite.

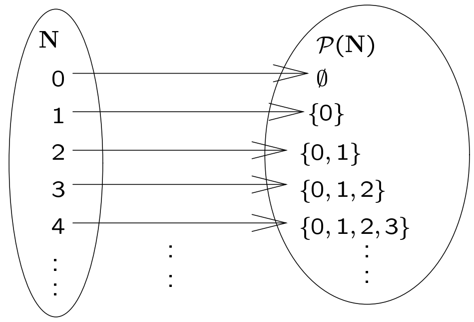
Q Are there more infinite sets?

Example: Infinite Set



We say that the natural numbers, $\mathbf{N}$, and all equivalent sets have cardinality $\aleph_0$ (aleph zero).

Q Are there sets with higher cardinality than $\aleph_0$? (i.e. are there sets that are larger than the set of natural numbers?)



$|\mathcal{P}(\mathbf{N})| = \aleph_0$?

Theorem The powerset of a set A, has higher cardinality than the set itself, i.e. $|A| < |\mathcal{P}(A)|$.

Proof Either $|A| = |\mathcal{P}(A)|$ or $|A| < |\mathcal{P}(A)|$. ($|A| > |\mathcal{P}(A)|$ is not possible since there is a one-to-one correspondence between A and the sets with exactly one element, e.g. $f(a) = \{a\}$.)
Assume that $F : A \to \mathcal{P}(A)$ is a one-to-one correspondence.
Let $B = \{x \in A \mid x \notin F(x)\}$. The we know that $B \in \mathcal{P}(A)$.
Since F is a one-to-one correspondence, there must exist a y such that $F(y) = B$. Now, does $y \in B$ hold?
1) If $y \notin B$ then $y \notin F(y)$ $(= B)$, and therefore $y \in B$. Contradiction.
2) If $y \in B$ then $y \in F(y)$ $(= B)$, and therefore $y \notin B$. Contradiction.

Thus, there can be no one-to-one correspondence $F : A \to \mathcal{P}(A)$ and, therefore $|A| \neq |\mathcal{P}(N)|$. The only remaining possibility is that $|A| < |\mathcal{P}(A)|$.

So, $|\mathbf{N}| < |\mathcal{P}(\mathbf{N})|$!

We write $|\mathcal{P}(\mathbf{N})| = 2^{\aleph_0}$.

Other examples: $\mathbf{R} = \{\text{real number}\}$, i.e. the set of all numbers that can be written with a possibly infinite sequence of decimals. The set of is said to have cardinality $\aleph_1$

A finite set, or with cardinality $\aleph_0$ is called countable.

An Open Problem in Mathematics: The Continuum Hypothesis

Does it exists a set, M, with cardinality strictly between $\aleph_0$ and $\aleph_1$, i.e. $\aleph_0 < |M| < \aleph_1$?

An unsolved problem for 100 years!

A

- Two sets are equally large if there exists a one-to-one correspondence between them.
- The size (cardinality $|M|$) of a finite set, M, is the number of elements in t M.
- An infinite set is a set which is equivalent to a proper subset of itself.
  The cardinality of the set of natural numbers is denoted $\aleph_0$. There exist sets with higher cardinality, e.g. $\mathcal{P}(\mathbf{N})$.

Natural Numbers - An Infinite Set

We will now examine how to

- define every natural number in a purely set theoretic way. Set theory is the basis for all mathematics!
- give a finite description of an infinite set.

Natural Numbers Cont'd

Problem: How can we represent every natural number only using sets?

Idea: Let $0 =_{\text{def}} \emptyset$ (no elements).

Idea: Let $1 =_{\text{def}} \{\emptyset\}$ (1 element).

Note! $\emptyset \neq \{\emptyset\}$, i.e. $0 \neq 1$.
$\emptyset \subset \{\emptyset\}$, some similarity to $0 < 1$!

Natural Numbers Cont'd

Idea: Let $2 =_{\text{def}} \{\emptyset, \emptyset\}$ (2 elements).

Problem: $2 = \{\emptyset, \emptyset\} = \{\emptyset\} = 1$, i.e. $2 = 1$! Not good...
We want every number to be unique!

Another idea: Let $2 =_{\text{def}} \{0, 1\} = \{\emptyset, \{\emptyset\}\}$ (2 elements).

Now note: $2 = \{\emptyset, \{\emptyset\}\} \neq \{\emptyset\} = 1$, and, thus, $2 \neq 1$.

Natural Numbers Cont'd

Note:
$\emptyset \subset \{\emptyset, \{\emptyset\}\}$, makes $0 < 2$.
$\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$, makes $1 < 2$.

I.e. $\subset$ still works as $<$!

Natural Numbers Cont'd

We continue to construct every natural number from the previous numbers.

$$
\begin{aligned}
3 &=_{\text{def}} \{0, 1, 2\} = \{\emptyset,\ \{\emptyset\},\ \{\emptyset, \{\emptyset\}\}\ \} && (1)\\
4 &=_{\text{def}} \{0, 1, 2, 3\} && (2)\\
&= \{\emptyset,\ \{\emptyset\},\ \{\emptyset, \{\emptyset\}\},\ \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\ \} && (3)\\
&\quad ...\\
n &=_{\text{def}} \{0, 1, 2, \ldots, n-1\} && (4)
\end{aligned}
$$

It is getting complicated to enumerate the natural numbers in this way...

Natural Numbers Cont'd

A more compact description based on the same idea:

Define the *successor function* $S(n) = n \cup \{n\}$.
For a given number n, S yields the number that immediately comes after n, i.e. $S(n) = n + 1$.

We write n as $\underbrace{S(..S}_{n}(0)..)$.

Note that $n + 1 = \{0, 1, \ldots, n\}$ still holds.

Natural Numbers Cont'd

$S(n) = n \cup \{n\}$. We write n as $\underbrace{S(..S}_{n}(0)..)$.
Ett exempel:

$$
\begin{aligned}
3 &= S(S(S(0))) && (5)\\
&= S(S(S(\emptyset))) && (6)\\
&= S(S(\emptyset \cup \{\emptyset\})) && (7)\\
&= S(S(\{\emptyset\})) && (8)\\
&= S(\{\emptyset\} \cup \{\{\emptyset\}\}) && (9)\\
&= S(\{\emptyset, \{\emptyset\}\}) && (10)\\
&= \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} && (11)\\
&= \{\emptyset,\ \{\emptyset\},\ \{\emptyset, \{\emptyset\}\}\} && (12)
\end{aligned}
$$

Natural Numbers Cont'd

To make life a little easier we, e.g., write 22 instead of
$S(S(S(S(S(S(S(S(S(S(S(S(S(S(S(S(S(S(S(S(S(S(0))))))))))))))))))))))$
or
$\{\emptyset,\ \{\emptyset\},\ \{\emptyset,\{\emptyset\}\},\ \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\},\ \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\},\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}$
(pust...).

Natural Numbers Cont'd

Now we can define the set $\mathbf{N}$ of all natural numbers:

- $0 \in \mathbf{N}$.
- If $x \in \mathbf{N}$ then $S(x) \in \mathbf{N}$.
- Nothing else belongs to $\mathbf{N}$.

This is an inductive definition. We will discuss this more later on.

Relations on the Natural Numbers

Equality between numbers is defined in term of set equality.

The ordering "less than" $<$ is defined as proper subset $\subset$.
E.g. $1 < 2$ because $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

Functions on the Natural Numbers

The operation $+$ is defined as:

$$(1)\quad x + 0 = x \tag{13}$$
$$(2)\quad x + S(y) = S(x) + y \tag{14}$$

Example $(1+2)$:

$$\begin{aligned} S(0) + S(S(0)) &= S(S(0)) + S(0) && (15)\\ &= S(S(S(0)) + 0 && (16)\\ &= \underbrace{S(S(S(0)))}_{=3} && (17) \end{aligned}$$

Functions on the Natural Numbers Cont'd

The operation $\cdot$ is defined as:

$$(1)\quad x \cdot 0 = 0 \tag{18}$$
$$(2)\quad x \cdot S(0) = x \tag{19}$$
$$(3)\quad x \cdot S(S(y)) = (x \cdot S(y)) + x \tag{20}$$

Example $(2 \cdot 2)$:

$$\begin{aligned} S(S(0)) \cdot S(S(0)) &= (S(S(0)) \cdot S(0)) + S(S(0)) && (21)\\ &= S(S(0)) + S(S(0)) && (22)\\ \ldots &= \underbrace{S(S(S(S(0))))}_{=4} && (23) \end{aligned}$$

The Set of Natural Numbers N

Inductive Definition of an Infinite Set.
$0 = \emptyset$ and $S(n) = n \cup \{n\}$.
(1) $0 \in \mathbf{N}$.
(2) If $x \in \mathbf{N}$ then $S(x) \in \mathbf{N}$.
(3) Nothing else belong to $\mathbf{N}$.
Definition of $=$ and $<$.
Definition of $+$ and $\cdot$.

Summary

- Finite and infinite sets.
- Equally large = equivalent iff there exists a one-to-one correspondence.
- Size = Cardinalitet ($|M|$).
  Finite set $\Rightarrow$ Cardinalitet = the number of elements.
- A set is infinite iff it is equivalent to a proper subset of itself.
- Infinite sets may have different cardinality (have different size), e.g. $\aleph_0$ for $\mathbf{N}$ and $\aleph_1$ for the set of reals.