Take-Home Exercises 4(4)
Discrete Mathematics (HKGAB4) 1998: Infinite Sets, Induction, and Recursion

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1. Show that the two sets $A = \{a, b, c, d, e\}$ and $B = \{f, g, h, i, j\}$ are equivalent (use the definition of equivalent sets).

2. Show that the set of negative whole numbers, $\mathbf{Z}_{-} = \{-1, -2, -3, \ldots\}$, is infinite (use the definition of infinite sets).

3. Define the sets EO of even negative, and odd positive numbers inductively, i.e. $EO = \{1, -2, 3, -4, 5, -6, \ldots\}$.

4. A list is a finite sequence of elements, e.g. $[1, 2, 5, 7]$. We assume that all elements belong to $\mathbf{N}$. We write the empty list as $[]$.

   (a) Define the set $List(\mathbf{N})$ of all lists. Use the operator $|$ that, given an element, e.g. 3, and a list, e.g. $[2, 4, 7]$, creates a new list $3|[2, 4, 7] = [3, 2, 4, 7]$. Generally: If x is an element and L is a list we write $x|L$ to denote the list we get by putting x in the first position of L. Think about: What list should be used as base?

   (b) Define a recursive function $Ld : List(\mathbf{N}) \to \mathbf{N}$ that computes the number of elements in a list.

5. Show with an induction proof that the sum of the k first odd positive whole numbers is k^2. For example, we have $1 + 3 + 5 = 9 = 3^2$. Hint: An odd positive numbers can be written $(2i - 1)$, where i is a whole number, e.g. $3 = (2 \cdot 2 - 1)$. We can write what we want to show as $(2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + \ldots + (2 \cdot k - 1) = k^2$. As an induction hypothesis you can use $(2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + \ldots + (2 \cdot k - 1) = k^2$, and show that $(2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + \ldots + (2 \cdot k - 1) + (2 \cdot (k+1) - 1) = (k+1)^2$ in the induction step. Remember that $(k + 1)^2 = k^2 + 2k + 1$.

6. (a) Define a recursive function that computes $f(n) = 2^n$, $n \in \mathbf{N}$ by using multiplication.

(b) Use an induction proof to show that $2^n < n!$ for all $n > 3,\ n \in \mathbf{N}$.